SOLUTIONS FOR

aab Update 03.153.s19.pdf

1. We are given

$$|\measuredangle ABC| + |\measuredangle ADC| = 180$$
$$|\measuredangle BAD| + |\measuredangle BCD| = 180$$

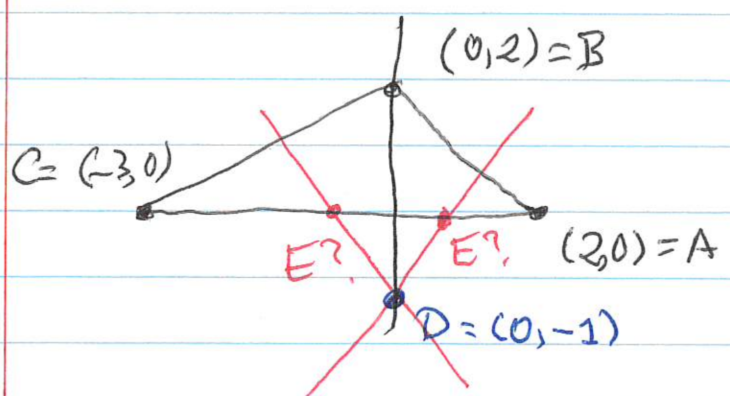
(inscribed in circle)

$$|\measuredangle ABC| = |\measuredangle ADC|$$
$$|\measuredangle BCD| = |\measuredangle BAD|$$

(parallelogram)

Hence $2|\measuredangle ABC| = 180 = 2|\measuredangle BCD|$, so $|\measuredangle ABC| = |\measuredangle BCD| = 90$. Now use the last two equations to conclude that $|\measuredangle BAD| = 90 = |\measuredangle ADC|$. ■

2.



$E = (q, 0)$

Find the equation defining DE:

$$y = \frac{x}{q} - 1 \quad \text{if } q \neq 0$$

$$x = 0 \quad \text{if } q = 0.$$

In the second case DE passes through B, so DE meets both [BA] and [BC]. Henceforth assume $q \neq 0$.

Case 1 $q > 0$. Find the point where DE meets AB. The latter has eqn $x + y = 2$.

$$\text{Solve} \begin{cases} y = \frac{x}{q} - 1 \\ y = 2 - x \end{cases} \qquad x = \frac{3q}{q+1}, \quad y = \frac{2-q}{q+1}$$

Since $0 < q < 2$, we have $y > 0$, so the intersection point lies on [AB]. ∎

Case 2 $q < 0$. The line BC has equation

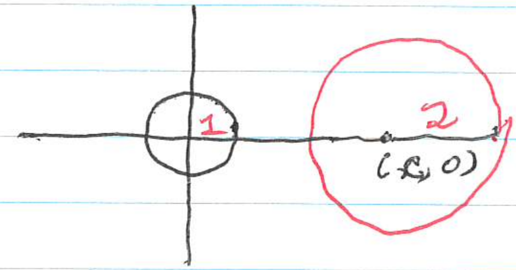
$$\frac{y}{2} - \frac{x}{3} = 1$$ so we need to solve

the system consisting of this eqn and $y = \frac{x}{q} - 1$.

Writing the second equa as $1 = \frac{x}{q} - y$ we get

$\frac{3y}{2} = \frac{q+3}{3q} x$, and we also get $3 = \left(\frac{1}{q} - \frac{2}{3}\right) x$.

Since $0 > q > 3$ the coefficient of x is negative and hence $x < 0$ ($3 = \text{neg} \times \text{neg}$). But now the same considerations for $y = \frac{2}{3} \frac{q+3}{3q} x$ imply $y > 0$. $\left(\begin{matrix} q+3>0 \\ 3q<0 \end{matrix}\right)$. Hence point $\in$ [BC]. ∎

3. 

One way to analyze this is to think of the red circle moving along the x-axis and finding the intersection points as the circle moves.

If $c > 3$ or $c < -3$, no points in common

If $c = \pm 3$, one point in common.

If $1 < |c| < 3$, two points in common.

If $|c| = 1$ one point in common.

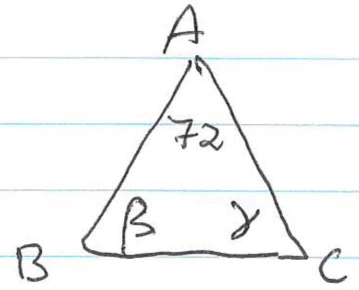
If $|c| < 1$ no points in common.

You should check this by solving the system $x^2 + y^2 = 1$ $(x-c)^2 + y^2 = 4$ in each of the five cases. ∎

4. ~~There are two~~ cases.

~~Case 1 [AB] is not one of the sides with equal length. Then $|\angle ABC| = |\angle ACB|$ since $|AB| = |BC|$. Hence $72 + 2|\angle ABC| = 180$, so that $|\angle ABC| = 54°$.~~

~~Case 2 $|AB| = |AC|$~~

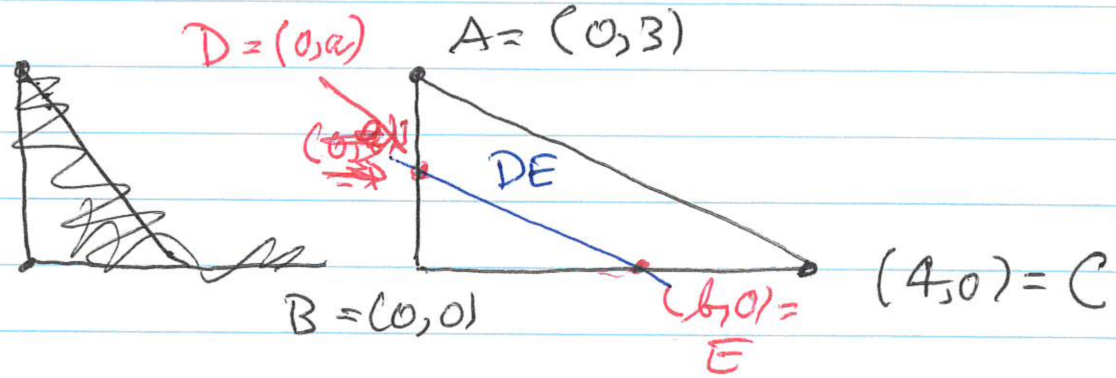
4. 

There are three cases depending upon whether $|AB| = |AC|$, $|AB| = |BC|$ or $|AC| = |BC|$. Note that all 3 sides cannot have the same length, for then all angles would be $60°$ angles.

Case 1 $|AB| = |AC|$. Then $\beta = \gamma$ by the Isosceles Triangle Thm., so $72 + \beta + \gamma = 72 + 2\beta = 180°$, so that $\beta = 54°$.

Case 2 $|AB| = |BC|$. Then $\gamma = 72°$ and $144 + \beta = 180$, so that $\beta = 36°$.

Case 3 $|AC| = |BC|$. Then $\beta = 72°$. $\blacksquare$

5. 

We want to show that DE does not meet [AC] (but it may meet the line AC!).

Following the hint, we need to solve the system

$$\frac{x}{4} + \frac{y}{3} = 1$$

$$\frac{x}{b} + \frac{y}{a} = 1 \qquad \begin{array}{l} 0 < a < 3 \\ 0 < b < 4 \end{array}$$

and show that if (u_0, v_0) is ~~∄~~ a solution then $0 \leq v_0 \leq 3$ does $\boxed{\text{NOT}}$ hold.

Let's use determinants (Cramer's Rule):

$$x = \frac{\begin{vmatrix} 1 & \frac{1}{3} \\ 1 & \frac{1}{a} \end{vmatrix}}{\begin{vmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{1}{b} & \frac{1}{a} \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} \frac{1}{4} & 1 \\ \frac{1}{b} & 1 \end{vmatrix}}{\begin{vmatrix} \frac{1}{4} & \frac{1}{3} \\ \frac{1}{b} & \frac{1}{a} \end{vmatrix}}$$

This is valid if the denominator $\neq 0$. But if the latter happens, then $\frac{1}{4a} = \frac{1}{3b}$, so that $\frac{3}{4} = \frac{a}{b}$ or $a = 3r$, $b = 4r$ $(r \neq 0)$. In this case the two lines are parallel so there is no solution and we are done.

Now suppose the determinant $\neq 0$.
denominator

Then we have

$$x = \frac{\frac{1}{a} - \frac{1}{3}}{\frac{1}{4a} - \frac{1}{3b}} \qquad y = \frac{\frac{1}{4} - \frac{1}{b}}{\frac{1}{4a} - \frac{1}{3b}}$$

The ~~numer~~ denominators are identical, but $b < 4$ and $a < 3$ imply the numerators have opposite signs. Therefore $\boxed{x \text{ and } y \text{ have opposite signs.}}$ If $(x,y) \in [AC]$ then $x \geq 0$ and $y \geq 0$. Since one of x, y is positive and the other is negative, it follows that the common point (x,y) cannot lie on the hypotenuse $[AC]$. ∎