Math 153
Spring 2019

UPDATED GENERAL INFORMATION — MAY 23, 2019

The third quiz

This will take place on Thursday, May 30, and it will cover material from Units 6X through 9 (however, there were no assigned problems for Chapter 8); note that although 6X was listed in the course coverage for the second quiz, the quiz contained nothing from that unit.

Here are some practice problems. Solutions will be forthcoming.

1. (*a*) If p is a prime and $0 < k < p$ explain why the binomial coefficient

$$\binom{p}{k} \quad = \quad \frac{p!}{k!(p-k)!}$$

is divisible by p. [*Hint:* Look for factors of p in the numerator and denominator.]

(*b*) Suppose that a and b are integers such that $a \equiv b \bmod(p)$. Show that $a^p \equiv b^p \bmod(p^2)$. [*Hint:* : Write $b = a + kp$.]

2. (*a*) Suppose that $n > 1$ is an integer and r is another integer such that $r \not\equiv 0, 1 \bmod(n)$ and $r^2 \equiv r \bmod(n)$. Prove that n is not prime. [*Hint:* Use the fact that if n and r are relatively prime then there is some integer q such that $qr \equiv 1 \bmod(n)$.]

(*b*) Give an examnle of integers a and n such that $a^n \not\equiv a \bmod(n)$. Note that by the Little Fermat Theorem n cannot be a prime number.

3. (*a*) Let $n > 1$ be an integer. Explain why $k^2 \equiv (n-k)^2 \bmod(n)$ for all k.

(*b*) Find all integers a such that $0 \leq a \leq 10$ such that $a \equiv b^2 \bmod(11)$ for some integer b. [*Hint:* Part (*a*) may help reduce the amount of calculation needed.]

(*c*) Find all integers a such that $0 \leq a \leq 12$ such that $a \equiv b^2 \bmod(13)$ for some integer b.

The next two problems involve some numerical issues which arise from the Cubic Formula in Chapter 9 of the course notes.

4. The Cubic Formula shows that one root of the polynomial $x^3 - 3x + 1 = 0$ has the form

$$\sqrt[3]{\cos(2\pi/3) + i\sin(2\pi/3)} \ + \ \sqrt[3]{\cos(2\pi/3) - i\sin(2\pi/3)} \ .$$

Express this as a real number; your answer should have the form $\cos\theta$ for an explicit value of θ. [*Hint:* $e^{i\alpha} = \cos\alpha + i\sin\alpha$.]

5. The Cubic Formula shows that one root of the polynomial $x^3 + x^2 - 2 = 0$ has the form

$$\frac{1}{3}\left(\sqrt[3]{26 + 15\sqrt{3}} \ + \ \sqrt[3]{26 - 15\sqrt{3}} \ - \ 1\right) \ .$$

Using Bombelli's methods, show that this expression is a single digit positive integer. The crucial step is to show that the expressions under the cube root signs are equal to $a \pm b\sqrt{3}$ for two single digit integers a and b.