Sexagesimal expansions of numbers between 0 and 1

It might be helpful to recall a basic rule for determining a (base 10) decimal expansion of a real number.

Formula. *Suppose that x is a real number strictly between 0 and 1 whose decimal expansion has the form $0.a_1a_2a_3a_4\cdots$ where the a_j are integers in the set $\{0,1,2,3,4,5,6,7,8,9\}$; in the ambiguous case where x is a finite decimal fraction, take the expansion which has infinitely many zeros. Then a_1 is the greatest integer $\leq 10x$.*

NOTATION. We shall lett $[y]$ be the greatest integer $\leq y$.

We can use this to find each a_j recursively as follows: Let $r_0 = x$, so that $10x = 10r_0 = [10r_0] + r_1$, where $0 \leq r_1 < 1$. Since our original decimal expansion was uniquely chosen and does not end with an infinite sequence of 9's, it follows that $[10r_0] = a_1$ and r_1 has decimal expansion $0.a_2a_3a_4\cdots$. We may continue in this fashion, defining $r_j \in [0,1)$ recursively by

$$r_j \quad = \quad 10r_{j-1} \; - \; [10r_{j-1}]$$

and at each step we have $a_{j-1} = [10r_{j-1}]$.

The same procedure works if we want expansions with respect to a base B instead of 10, where B is any positive integer strictly greater than 1. In particular, if we want to find the base 60 expansions used by the Babylonians, we take $B = 60$.

SIMPLE EXAMPLE. Consider the fraction 1/6. We could just say $1/6 = 10/60$ to see that the base 60 expansion

$$\frac{1}{6} \quad = \quad 0.b_1;b_2;b_3;b_4\cdots$$

(where the b_j are integers between 0 and 59) is equal to $0.10;0;0;\cdots$; we are using semicolons to separate the values of the terms here and distinguish the expression from a decimal expansion). However, we can also verifying this using the base 60 version of the preceding rule as follows: If $x = 1/6 = r_0$, then $60x = 6 = [60r_0]$. Therefore $b_1 = 60$ and $r_1 = 60 - 60 = 0$. But now $b_2 = [60r_1] = [0] = 0$, and it follows that $b_2 = 0 = r_2$. Continuing in this manner, we see that $b_3 = 0 = r_3$ and similarly $b_j = 0 = r_j$ for all $j \geq 3$. Here is a slightly less trivial example.

PROBLEM. *Convert the ordinary fraction 1/40 to sexagesimal notation.*

Before solving this, we note it is equivalent to a questiion which can be asked at the elementary school level: *How many minutes and seconds are there in 1/40 of an hour?*

SOLUTION. Here $x = r_0 = 1/40$ and hence we have $b_1 = [60/40] = 1$ and

$$r_1 \quad = \quad \frac{60}{40} - \left[\frac{60}{40}\right] \quad = \quad \frac{1}{2}$$

and at the next step we have $b_2 = [60r_1] = 30$ and

$$r_2 \quad = \quad 60r_1 \; - \; [60r_1] \quad = \quad 30 \; - \; 30 \quad = \quad 0\,.$$

As in the first example, since the remainder r_2 is zero it follows that all the subsequent expansion terms b_k and r_k must also be zero.

Therefore the base 60 expansion of 1/40 is given by $0.1;30[;0;0$ *etc.*]. Returning to the elementary reformulation, the answer simply means that 1/40 of an hour is equal to 1 minute and 30 seconds.

TRY THIS: *Convert the ordinary fraction 1/50 to sexagesimal notation.*

The file `base60change2.pdf` discusses further examples.