1. Mathematics in the earliest civilizations

(Burton, **1.1 – 1.3, 2.2 – 2.5**)

> If you really want to hear about it, the first thing you'll probably want to know is where I was born, and … how my parents were occupied and all before they had me, and all that … but I don't feel like going into it, if you want to know the truth.
>
> J. D. Salinger (1919 – 2010), ***The Catcher in the Rye***

Opinions will probably always differ about the "correct" beginning for the prehistory of mathematics, but a few things are clear. Both archaeology and anthropology show that most if not all human cultures have had at least some crude concepts of numbers, with some of the earliest archaeological evidence scientifically dated around **30,000** years ago. Numerous archaeological discoveries also indicate that many prehistoric cultures had discovered that counting larger quantities was easier with some means of grouping together fixed numbers of objects. For example, twelve stones could be arranged in two groups of five and one group of two, and similarly for other numbers one can count off groups of five until there are less than five items left. Such arrangements are the first step in the development of a number system.

Although the study of rudimentary number concepts in prehistoric and other cultures is potentially an interesting subject, for our purposes it will be best simply to recognize the near – universal awareness of the number concept as given and to move on to the development of mathematics within the ancient civilizations that emerged about **5000** years ago. In this unit we shall focus on two civilizations that have had particularly strong impacts on mathematics as we know it today; namely, the ancient Egyptian and Mesopotamian civilizations. Extensive information on numbering systems in other cultures is contained in the following reference:

> **G. Ifrah**, ***The universal history of numbers. From prehistory to the invention of the computer.*** (Translated from the 1994 French original by D. Bellos, E. F. Harding, S. Wood and I. Monk.) *Wiley, New York,* 2000. **ISBN:** 0–471–37568–3.

Existing records of ancient civilizations are often very spotty in many respects, with very substantial information on some matters and little if anything on others. Therefore any attempt to discuss mathematics in ancient civilizations must recognize that one can only discuss what is known from currently existing evidence and accept that much of these cultures' mathematics has been lost with the passage of time. However, we can very safely conclude that such cultures were quite proficient in some aspects of practical mathematics, for otherwise many of the spectacular engineering achievements of ancient cultures would have been difficult to plan and impossible to complete. **ALL** statements made about pre – Greek mathematics must be viewed in this light, and for the advanced civilizations elsewhere of the world (China and India, in particular) the unevenness of evidence extends to even later periods of time.

Expressions for whole numbers in Egypt and Mesopotamia

In most but not all cases, the development of written records is closely linked to the birth of a civilization, and many such records are basically numerical. Therefore we have some understanding of the sorts of numbering systems used by most of the ancient civilizations. In most cases it is apparent that these civilizations had also discovered the concept of fractions and had devised methods for expressing them.

One extremely noteworthy point is that different civilizations often took quite different approaches to the problem of setting up workable number systems, and this applies particularly to fractions. Perhaps the simplest question about number systems concerns the choices for grouping numbers. The numbers **5** and **10** usually had some particular significance. For example, Egyptian hieratic writing – which was a simplification of the older hieroglyphics – had separate symbols for **1** through **9,** multiples of **10** up to **90,** multiples of **100** through **900,** and multiples of **1000** through **9000;** it should also be noted that the earlier hieroglyphic Egyptian writing included symbols for powers of **10** up to ten million). A number like **256** would then be represented by the symbols for **200, 50** and **6;** this is totally analogous to the Roman numeral expression for **256** as **CCLVI.** Numerous other cultures had similar systems; in particular, in classical Greek civilization the Greek language used letters of the ancient alphabet to denote numbers from **1** to **9,** **10** to **90** and **100** to **900** in exactly the same fashion (see the chart on page 4 of the course directory file **http://math.ucr.edu/~res/math153/history06X.pdf**).

Although **10** has played a key role in most number systems, there have been some notable exceptions, and traces of some are still highly visible in today's world (for example, the French word ***quatre – vingts***, or "four twenties" for **80,** and comparable words in certain other languages). The Mayan civilization placed particular emphasis on the numbers **5** and **20.** Roman numerals indicate a special role for **5** and **10.** However, the Sumerians in Mesopotamia developed the most extraordinary alternative during the 3rd millennium B.C.E.. They used a ***sexagesimal*** (or base **60**) system that we still use today for telling time and some angle measurements: One degree or hour has sixty minutes, and one minute has sixty seconds. The Mesopotamian notation for numbers from **1** to **59** is strikingly similar to the notation we use today. In particular, if n is a positive integer less than **60** and we write

$$n = 10p + q, \quad \text{where} \quad 0 \leq p \leq 5 \quad \text{and} \quad 1 \leq q \leq 9$$

then n was written as a combination of p thick horizontal strokes and q thin vertical strokes; there is a table of the Babylonian numerals from **1** to **59** (the notion of zero did not exist then) in **http://math.ucr.edu/~res/math153-2020/week2/unit1/cuneiform.jpg**. Much like our modern number system, larger positive integers were expressed in a form like

$$a_0 + a_1 \times 60 + a_2 \times 60^2 + a_3 \times 60^3 + \ldots + a_N \times 60^N$$

where each a_j is a nonnegative integer that is less than **60,** but at first there were problems when one or more of the numbers a_j was equal to zero, and eventually place holders were used in positions where we would insert a zero today. However, as noted on page 25 of Burton, there is nothing to indicate that any such place holder was "regarded … as a number by itself that could ever be used for computational purposes."

Egyptian fractions

The differences between the Egyptian and Mesopotamian representations for fractions were far more significant. For reasons that are not completely understood, the Egyptians expressed virtually all fractions as finite sums of ordinary reciprocals or *unit fractions* of the form

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + (etc.)$$

where ***no denominator appears more than once in the expansion.*** Of course this restriction leads to complicated expressions even for many fairly simple fractions, and the discussion and tables on page 40 of Burton give expansions for a long list of fractions with small denominators.

As noted on page 43 of Burton, every rational number between 0 and 1 has an ***Egyptian fraction*** expansion of this type. Perhaps the most widely known method for finding such expressions is the so – called ***Greedy Algorithm*** which is due to the 13th century Italian mathematician Leonardo of Pisa (*c.* 1170 – 1250 A.D., better known as ***Fibonacci***; more will be said about him later). A description of this method and a proof that it works are reproduced below, this account is slightly adapted from the online site

<http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fractions/egyptian.html#Fibgreedy>

which also has information on several other aspects of Egyptian fractions.

One possible use of Egyptian fraction expansions. Before proving the general result on Egyptian fraction expansions, it seems worthwhile to make some general comments and to indicate some "practical" sorts of problems in which such expansions are useful. Such expansions were often needed in order to divide inheritances or other property. Here is a simplified example:

Suppose we have 7 loaves of bread that are to be divided among 10 individuals. Find a "fair" way of splitting the loaves evenly.

Obviously everyone should end up with $7/10$ of a loaf of bread. However, it is likely that there will be complaints if, say, the first person is given $7/10$ of the first loaf, the second is given $3/10$ of the first loaf and $2/5$ of the second, the third is given the remaining $3/5$ of the second loaf and $1/10$ of the third, and so on. There will be fewer reasons for complaint if everyone receives pieces of the same size(s).

How can this be done using Egyptian fraction expansions? First of all, we have

$$7/10 = 1/2 + 1/5$$

and therefore we have that $7 = (1/2) \times 10 + (1/5) \times 10.$ In other words, we can give everyone the same sized pieces if we give each person a half of one loaf and a fifth of another loaf. This can be done by cutting five of the loaves into halves and the remaining two into fifths (see

http://math.ucr.edu/~res/math153-2020/week2/unit1/seven-tenths.pdf

for an illustration).

Finding Egyptian fraction expansions. Well before Fibonacci's time in the early 13^th^ century A.D., more efficient methods had supplanted Egyptian fractions for many everyday computations, and in fact Fibonacci himself often used other representations of fractions. However, they continued to see limited use until the late Middle Ages, and thus there was enough remaining interest for a leading mathematician of the day to give a logically rigorous procedure for finding such expansions.

There is still some remaining interest in questions about Egyptian fractions for a variety of reasons. As an indication of how Egyptian fraction expansions have continued to generate mathematical interest we mention a simply stated unsolved problem raised by the celebrated mathematician P. Erdős (AIR – desh, 1913 – 1996) and E. G. Straus (1922 – 1983): *Suppose that* n *is an odd number which is at least* 5. *Is it always possible to write the fraction* $5/n$ *as a sum of three unit fractions*?

Notes. (1) There is a brief discussion of Erdős' unusual life and extensive work on page 747 of Burton.

(2) Some results on Egyptian fraction expansions that are related to the Erdős – Straus problem are discussed in **http://math.ucr.edu/~res/math153/history01a.pdf**.

(3) Egyptian fractions are a perfect example of a topic which originally arose in a "practical" context but has interesting features which have generated further study for its own sake. This phenomenon happens frequently in many areas of human activity, but it is particularly fundamental to mathematics, and it was already apparent in the **Rhind papyrus** from the 19th century B.C.E. (see pages 49 – 50 of Burton). The reasons for pursuing such mathematical questions for their own sake frequently go beyond simple curiosity and enjoyment. Very often a topic that originally seemed interesting in its own right will eventually be useful in a more serious mathematical inquiry. Further discussion of this appears in an article by D. Singmaster (*The unreasonable utility of recreational mathematics*), which is available at the following online site:

http://anduin.eldar.org/~problemi/singmast/ecmutil.html

A ***Google*** search for "mathematical puzzles" and "mathematical games" will produce an enormous number of sources for examples of such items.

Fibonacci's algorithm. We now return to a statement and proof of the Greedy Algorithm for expressing an arbitrary fraction as a sum of unit fractions.

FIBONACCI'S METHOD, A. K. A. THE GREEDY ALGORITHM: This method and a proof are given by Fibonacci in his book ***Liber Abaci*** which was written in or around 1202. The latter also contains a discussion of the so – called *Fibonacci Numbers* (incidentally, the latter had been considered by earlier mathematicians from India; we shall discuss this topic later). We begin our derivation of the Greedy Algorithm by noting that

- $T/B < 1$ and
- If $T = 1$, then the problem is solved since T/B is already a unit fraction, so we are interested in those fractions where $T > 1$.

The first step is to find the ***biggest unit fraction*** $1/k$ such that $T/B - 1/k > 0$, and hence the name ***Greedy Algorithm*** for this process.

The same process may now be applied to the remainder T/B $- 1/k$, and so on. We shall prove that this series of unit fractions always decreases, never repeats a fraction and eventually will stop. Such processes are now called ***algorithms*** and this is an example of a ***greedy algorithm*** since we (greedily) take the largest unit fraction we can and then repeat on the remainder.

Let's look at the specific example of $521/1050$ before we present the proof. Now $521/1050$ is less than one-half (since 521 is less than a half of 1050) but it is bigger than $1/3$. So the largest unit fraction we can take away from $521/1050$ is $1/3$:

$$521/1050 \quad = \quad 1/3 \ + \ \text{R}$$

What is the remainder? To find it we simply subtract one fraction from the other:

$$521/1050 \ - \ 1/3 \ = \ 57/350$$

So we repeat the process on $57/350$:

This time the largest unit fraction less than $57/350$ is $1/7$ and the remainder is $1/50$.

> How do we know the denominator is 7? Divide the bottom (larger) number, 350, by the top one, 57, and we get $6.14\ldots$. So we need a number larger than 6 (since we have $6 + 0.14 \ \ldots$) and of course the next one above 6 is 7.

Thus we have $521/1050 \ = \ 1/3 \ + \ 1/7 \ + \ 1/50$. The sequence of remainders is important in the formal proof that we do not have to keep on doing this forever for some fractions T/B:

$$521/1050, \quad 57/350, \quad 1/50$$

In particular, although the ***denominators*** of the remainders are getting larger, the important fact that is true in ***<u>all</u>*** *cases* is that ***the numerator of the remainder is getting smaller***. If it keeps decreasing, then it must eventually reach 1 and at this stage the process stops.

A PROOF THAT THE PROCESS <u>ALWAYS</u> TERMINATES AFTER FINITELY MANY STEPS:

Now let's see how we can show this is true for all fractions T/B. We want

$$\text{T/B} \quad = \quad 1/u_1 \ + \ 1/u_2 \ + \ \ldots \ + \ 1/u_n$$

where $u_1 \ < \ u_2 \ < \ \ldots \ < \ u_n$. Also, we are choosing the largest u_k at each stage.

<u>What does this mean?</u> It means that $1/u_1 \ < \ \text{T/B}$, but also that $1/u_1$ is the ***largest*** such fraction. For instance, we found that $1/3$ was the largest unit fraction less than $521/1050$. This means that $1/2$ must be ***greater*** than $521/1050$.

In general, if $1/u_1$ is the largest unit fraction less than T/B then

$$1/(u_{11} - 1) \ > \ \text{T/B}.$$

Since $\text{T} \ > \ 1$, neither $1/u_1$ nor $1/(u_1 - 1)$ is equal to T/B. What is the remainder? It is

$$(T/B) - (1/u_1) = (T \cdot u_1 - B)/(B \cdot u_1).$$

Also, since $1/(u_1 - 1) > T/B$ we can multiply both sides by **B** to obtain

$$B/(u_1 - 1) > T.$$

Multiplying both sides by $u_1 - 1$ and expanding the brackets, then adding **T** and subtracting **B** on both sides yields the following:

$$B > T \cdot (u_1 - 1)$$
$$B > T \cdot u_1 - T$$
$$T > T \cdot u_1 - B$$

Now $T \cdot u_1 - B$ was the ***numerator of the remainder*** and we have just shown that ***it is smaller than the original numerator***, which is **T.** If the remainder, in its lowest terms, has a 1 on the top, we are finished. Otherwise, we can *repeat the process* on the remainder, which has a smaller denominator and so the remainder when we take off its largest unit fraction gets smaller still. Since **T** is a whole (positive) number, this process ***must*** inevitably terminate with a numerator of 1 at some stage.

This completes the proof of the following statements:

- For every fraction **T/B** between 0 and 1, there is always a ***finite*** list of unit fractions whose sum is equal to **T/B.**
- We can find such a sum by taking the largest unit fraction at each stage and repeating on the remainder (the ***Greedy Algorithm***).
- The unit fractions so chosen get smaller and smaller (and so all are unique).

DIFFERENT REPRESENTATIONS FOR THE SAME FRACTION: We obviously have $3/4 = 1/2 + 1/4$, but there are also other Egyptian fraction forms for $3/4$. For example we may also write $3/4$ as $1/2 + 1/5 + 1/20$ and $1/2 + 1/6 + 1/12$ and $1/2 + 1/7 + 1/14 + 1/28$. One could continue in this manner, but instead of doing so we shall discuss the underlying general principle:

<u>EVERY</u> FRACTION BETWEEN 0 AND 1 HAS <u>INFINITELY MANY</u> DIFFERENT EGYPTIAN FRACTION REPRESENTATIONS. We begin with an absolutely trivial observation:

$$1 = 1/2 + 1/3 + 1/6$$

By the Greedy Algorithm we know that every fraction **T/B** as above has at least one Egyptian fraction expression. This will be the first step of a proof by induction. Suppose that we have k distinct expressions as an Egyptian fraction for some positive integer k. To complete the inductive step we need to construct one more expression of this type for **T/B.**

Among all the k given Egyptian fraction expansions there is a minimal unit fraction summand $1/m$ (for each expansion there are only finitely many denominators, and within each one there is a minimum denominator). Choose one of these expansions

$$T/B = 1/u_1 + 1/u_2 + \ldots + 1/u_n$$

such that $u_n = m$. Then we may use the trivial identity above to obtain the following equation:

$$\mathbf{T/B} = 1/u_1 + 1/u_2 + \ldots + 1/u_{n-1} + 1/(2\cdot u_n) + 1/(3\cdot u_n) + 1/(6\cdot u_n)$$

We claim this expression is different from the each of k expressions that we started with. To see this, observe that the minimal unit fraction summand is equal to

$$1/(6\cdot u_n) = 1/(6\cdot m)$$

but the minimal unit fraction summand for each expression on the original list is at least $1/m$. Therefore we have obtained an expression that is not on the original list of k equations, which means there are at least $k+1$ different unit fraction expressions for **T/B** and thus completes the proof of the inductive step.

Clearly there are many other ways that new Egyptian fraction expressions can be created out of old ones; for example, in the preceding construction one can replace the identity $1 = 1/2 + 1/3 + 1/6$ by an arbitrary expansion of 1 as a sum of unit fractions.

In contrast, one can also prove that for a fixed positive integer **L** there are only finitely many ways of expressing **T/B** as a sum of at most **L** unit fractions. The proof of this uses mathematical induction and will be left to the exercises.

Is greed always good? This might have been true in part of the 1987 movie ***Wall Street***, but here is an example for which the Greedy Algorithm yields an Egyptian fraction expansion that is far from optimal. Specifically, the Greedy Algorithm yields the expansion

$$\frac{5}{121} = \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} + \frac{1}{873960180913} + \frac{1}{1527612795642093418846225},$$

but one also has the much better expansion

$$\frac{5}{121} = \frac{1}{33} + \frac{1}{121} + \frac{1}{363}.$$

This example is taken from **http://en.wikipedia.org/wiki/Egyptian_fraction**.

Current scholarship and Egyptian mathematics. There is a more comprehensive and extremely knowledgeable summary of our current understanding of Egyptian mathematics in the following article:

http://math.berkeley.edu/~wodzicki/160/MI_28_19.pdf

The following two books contain still more information:

A. Imhausen. Mathematics in ancient Egypt. A contextual history. *Princeton University Press, Princeton, NJ,* 2016.

M. Clagett. Ancient Egyptian science: a source book. Vol. 3. Ancient Egyptian mathematics. Memoirs of the American Philosophical Society, 232. *American Philosophical Society, Philadelphia, PA,* 1999.

Babylonian fractions

We begin with some standard terminology that may be misleading in some respects but is generally used and convenient. Mesopotamia had a succession of dominant kingdoms from the dawn of civilization through the Persian conquest in 538 B.C.E.. In discussions of the ancient mathematical history for this region, it is customary to use the term "Babylonian mathematics" for contributions during the period from approximately 2000 B.C.E. up to the end of the Seleucid Empire (the eastern remainder of Alexander the Great's empire) in 64 B.C.E. (as noted in the last part of Unit 4, during the latter both Greek and Babylonian mathematics actively studied questions related to astronomy). Most known examples of Babylonian mathematical writings are from the Old Babylonian period, beginning around 2000 B.C.E. and ending with the fall of Babylon to the Hittites, which conjecturally took place around 1595 B.C.E. (this may be off by up to 65 years in either direction).

As one might guess from the present day hierarchies of hours/minutes/seconds and degrees/minutes/seconds, Babylonian mathematics extended its sexagesimal numbering system to include representations for fractions. This crucial step gave Babylonian mathematics a huge computational advantage over Egyptian mathematics and allowed the Babylonians to make extremely accurate computations; their methods and results were unsurpassed by other civilizations until the Renaissance in Europe, nearly 19 centuries after the Persian conquest in the 6th century B.C.E..

Of course, it is not possible to express every rational number between $\mathbf{0}$ and $\mathbf{1}$ as a finitely terminating sexagesimal fraction. For example, the formula for $\mathbf{1/7}$ in sexagesimal form is

$$\mathbf{0;8,34,17,8,34,17,8,34,17,8,34,17,8,34,17,8,34,17,\underline{8,34,17,}}$$

where the underlining indicates an infinitely repeating periodic sequence of the given numbers. Babylonian mathematics almost always ignored such expressions, and they are conspicuously absent from the tables of fractions that are known to exist.

The following web sites are good sources of further material on Babylonian mathematics; the second indexes a particularly extensive collection of documents:

http://openlearn.open.ac.uk/course/view.php?id=3349

http://it.stlawu.edu/~dmelvill/mesomath/index.html

Most of our information about Egyptian mathematics comes from a fairly small number of manuscripts, but since Mesopotamian writings were preserved in baked clay tablets the sources of information for Babylonian mathematics are far more extensive, and many of these sources are still being translated and studied. Some recent work along these lines is described in the following article:

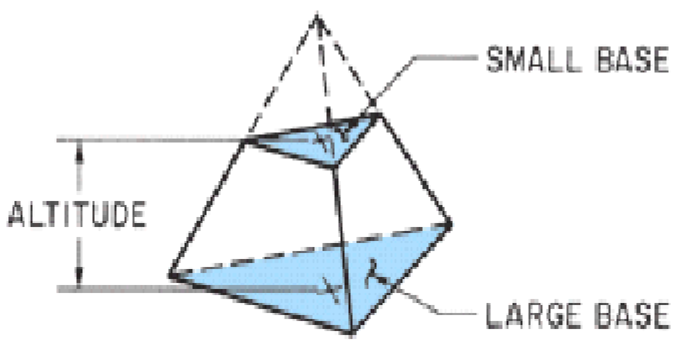
> C. Proust, Mathematics in Mesopotamia: From Elementary Education to Erudition, *The Institute Letter* (Institute for Advanced Study, Princeton USA) Spring 2010, p. 3 (2010); the latter is available online at
>
> **http://www.ias.edu/files/pdfs/publications/letter-2010-spring.pdf**.

Mathematical legacies of the early civilizations

Both the Egyptians and Babylonians were quite proficient in using arithmetic to solve everyday problems. Furthermore, both civilizations compiled substantial tables of values to be used when working problems. However, the emphasis was on specific problems rather than general principles. In particular, there is no evidence of proofs or comprehensive explanations of computational procedures, and the general discussions of procedures seemed to be directed at facilitating techniques rather than developing understanding. Given such an empirical approach, it was probably inevitable that there are mistakes in some of their procedures for finding answers to specific types of problems. Chapter 2 of Burton mentions one specific formula that both cultures got wrong: Given a "nice" quadrilateral **ABCD** in the plane such that the lengths of sides **AB**, **BC**, **CD** and **DA** are equal to a, b, c and d respectively, the writings of both cultures give the following formula for the area enclosed by **ABCD:**

area is supposedly equal to $(a+c)\cdot(b+d)/4$

Further discussion of this estimate for the area appears in Exercise 8 on pages 61–62 of Burton (see also Exercise 5 on page 80). In a few cases, one culture found the right formula for computing something while the other did not. A basic example of this sort involves the volume formula for a ***frustrum of a pyramid with a square base***; this is the object formed by taking a pyramid with a square base and slicing off the top along a plane that is parallel to the base (see page 56 of Burton). The drawing below illustrates a frustrum of a pyramid for which the base is triangular rather than square:



(***Source:*** **www.msubillings.edu/cotfaculty/pierce/classes/math122/ppt_from_book/Chapter%2029.ppt**)

The Egyptian formula was correct but the Babylonian one ($V = h(a+b)^2/4$) was not. An excellent interactive graphic for this figure and some interesting commentary on the Egyptian formula are available online at the site

http://mathworld.wolfram.com/TruncatedSquarePyramid.html

and the related site for the pyramidal frustrum listed there is also worth viewing. There is a simple derivation of the correct volume formula for the frustrum in the course directory file **http://math.ucr.edu/~res/math153/frustrum.pdf**.

Both the Egyptian and Mesopotamian civilizations developed numeration systems (with fractions) that were highly adequate for their purposes in many respects, but in each case there were difficulties with their approaches to fractions. In Egyptian mathematics, the most obvious problem concerned the clumsiness of the manner in which they wrote fractions, while in Babylonian mathematics there was the problem of dealing accurately with fractions $\mathbf{T}/\mathbf{B}$ for which the reduced version's (*i.e.*, **T** and **B** have no common factors except $+1$ and -1) denominator **B** is divisible by a prime greater than 5. More generally, the general lack of clear distinctions between approximate and actual values was a basic conceptual problem for both Egyptian and Babylonian mathematics.

Achievements and weaknesses of Egyptian mathematics

The Egyptian civilization was the first known to develop systematic calendars based upon lunar and solar cycles, maybe as early as the 5th millennium B. C. E., and some saw use into the Middle Ages due to their mathematical regularity. Some Egyptian numerical estimation procedures were quite good and elaborations of a few are still used today for some purposes (*e.g.*, the ***rule of false position*** for finding numerical values of solutions to equations; see **http://en.wikipedia.org/wiki/False_position_method** and **http://math.ucr.edu/~res/math153-2019/history01c.pdf** for descriptions), and Egyptian mathematics was clearly able to approximate square roots effectively. Archaeological evidence indicates a more extensive and accurate understanding of geometry than in Babylonian civilization. In particular, as noted before the Egyptians knew how to compute the volume of a truncated pyramid (see the bottom of page 56 in Burton) but the Babylonian formula was incorrect. Highly effective and precise surveying techniques also provide indirect evidence that the ancient Egyptians had a very firm command of geometry. However, although it is clear that Egyptian geometry provided valuable, perhaps indispensable, input to the later work of Greek geometers, evidence also suggests that the extent of these contributions are substantially less than classical Greek writers like the well – known historian Herodotus (*c.* 484 – 425 B. C. E.) have stated.

Egyptian arithmetic was based very strongly on addition and subtraction, and both multiplication and division were carried out by relatively awkward additive procedures. For example, if one wanted to multiply two positive integers **A** and **B**, this would begin with adding **A** to itself to form $2\mathbf{A}$, adding $2\mathbf{A}$ to itself to form $4\mathbf{A}$, and so on until one reaches the largest power of 2 such that $2^k < \mathbf{B}$. The next step would amount to finding the base two expansion of **B** as

$$e_0 1 + e_1 2 + \ldots + e_k 2^k \qquad (\textit{where each } e_j = 0 \textit{ or } 1)$$

and the final step would be to add all the numbers $2^p\mathbf{A}$ such that 2^p is a term which appears in the base two expansion of **B** (in other words, $e_p = 1$). Of course, computing this way is extremely clumsy by today's standards, but clearly the Egyptians were able to live with this and use it very successfully for many purposes.

One particularly noteworthy feature of Egyptian mathematics is that it apparently changed very little over a period of approximately two thousand years.

Achievements and weaknesses of Babylonian mathematics

The sexagesimal numeration system provided an extremely solid foundation for expressing a very large class of numbers and for doing all sorts of calculations to a very high degree of accuracy, and Babylonian mathematics realized this potential with an extensive collection of algorithms. In particular, their method for solving quadratic equations is equivalent to the quadratic formula that we still use in many situations. Babylonian mathematics was also quite proficient in solving large classes of cubic equations and systems of two equations in two unknowns. There were also other achievements related to algebra, but here we shall only mention the existence of evidence that Babylonian mathematics had some primitive understanding of trigonometric, exponential and logarithmic functions (however, we should stress that the extent of this understanding was extremely limited).

Although it appears that Egyptian mathematics understood at least some important special cases of the Pythagorean Theorem, the first known recognition of the Pythagorean Formula appears in Babylonian mathematics. As suggested by our earlier comments, there ware no attempts to justify the formula, but the Babylonians also studied integral solutions of the Pythagorean Equation

$$a^2 + b^2 = c^2$$

extensively; ***e.g.***, this is evident from the cuneiform tablet called **Plimpton 322**.

We have mentioned evidence that Egyptian geometry was relatively well – developed. It is more difficult to assess Babylonian achievements in this area, but the evidence does show a high level of proficiency in making geometric measurements of various kinds. Evidence indicates that Babylonian mathematicians were acquainted with a number of basic results in plane geometry.

Babylonian mathematics had a particularly strong impact on observational astronomy, including the traditional notation for locating heavenly bodies and prediction of solar and lunar eclipses.

Some weaknesses of Babylonian mathematics were already mentioned above. The lack of negative numbers is also worth mentioning, both for its own sake and its relation to another significant deficiency: *Babylonian mathematics only described one solution for a quadratic equation rather than two*. Of course, even if one ignores negative numbers there are many cases for which a quadratic equation has two positive roots, and recognizing only one root of a quadratic equations can clearly be a nontrivial omission in some situations.

We conclude this unit with an important caution about the preceding discussion. Although there is no direct evidence of formal proofs or comprehensive explanations of computational procedures, describing mathematics in early civilizations as a mainly utilitarian subject — which was devoid of logical structure, deductive proofs, generalizations or abstractions — is **almost surely** an oversimplification. Certainly none of these features appeared as explicitly or forcefully as they do in Greek mathematics, but the evidence also suggests some forms of these features must have been at least implicit. Among the arguments suggesting this are

- **(1)** descriptions of numerous problems that resemble each other quite closely and solving them by similar methods,
- **(2)** performing arithmetic operations with different types of measurements — for example, adding a length to an area.

As noted before, our information on both Egyptian and Mesopotamian mathematics is extremely spotty and there is plenty of room for speculation in many directions.

Addenda to this unit

There are five separate items. The first (**1A**) contains some results on Egyptian fractions with small numerators, and the second (**1B**) discusses base 60 expansions for unit fractions of the form $1/n$, where n is a positive integer. In the remaining files (**1C, 1Cprogram, 1D**) the method of false position is described and examples are worked out.