ANGLE BISECTION WITH STRAIGHTEDGE AND COMPASS

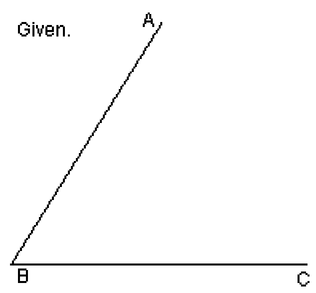
The discussion below is largely based upon material and drawings from the following site:

http://strader.cehd.tamu.edu/geometry/bisectangle1.0/bisectangle.html

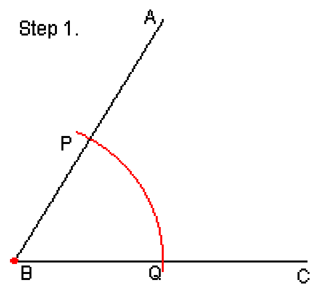
Angle Bisection Problem: To construct the angle bisector of an angle using an unmarked straightedge and collapsible compass.

Given an angle to bisect; for example, take $\angle$ **ABC.**

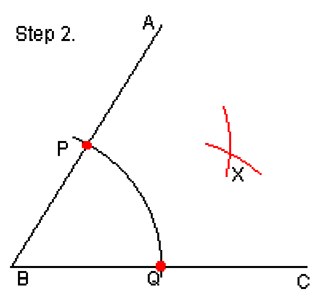
To construct a point **X** in the interior of the angle such that the ray **[BX** bisects the angle. In other words, |$\angle$ **ABX**| = |$\angle$ **XBC**|.



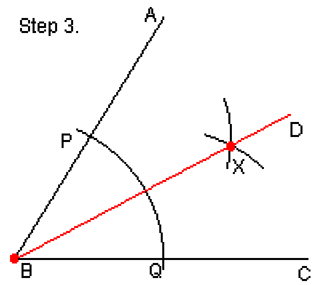
Step 1. Draw a circle that is centered at the vertex **B** of the angle. This circle can have a radius of any length, and it must intersect both sides of the angle. We shall call these intersection points **P** and **Q.** This provides a point on each line that is an equal distance from the vertex of the angle.



Step 2. Draw two more circles. The first will be centered at **P** and will pass through **Q,** while the other will be centered at **Q** and pass through **P.** A basic continuity result in geometry implies that the two circles meet in a pair of points, one on each side of the line **PQ** (see the footnote at the end of this document). Take **X** to be the intersection point which is **not** on the same side of **PQ** as **B.** Equivalently, choose **X** so that **X** and **B** lie on opposite sides of **PQ.**



Step 3. Draw a line through the vertex **B** and the constructed point **X.** We claim that the ray **[BX** will be the angle bisector**.**

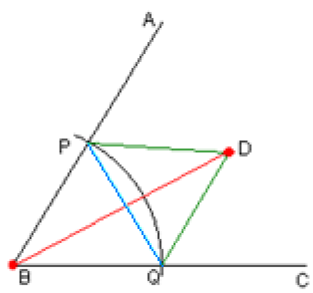


Proof that this construction yields the angle bisector.

We need to prove that **X** lies in the interior of $\angle$ **ABC** and that **|$\angle$ ABX|** = **|$\angle$ DBC|.** If we know these then we also have

$$|\angle \mathbf{ABC}| \;=\; |\angle \mathbf{ABX}| + |\angle \mathbf{XBC}| \;=\; 2|\angle \mathbf{ABX}| \;=\; 2|\angle \mathbf{XBC}|.$$

In most discussions of this construction the first statement is ignored**,** but we shall verify this fact because it is needed to prove the sum formula**.** By construction we know that **|BP|** = **|BQ|** and also **|PX|** = **|PQ|** = **|QX|.**



The equations **|BP|** = **|BQ|** and **|PD|** = **|QD|** imply that the line **BD** is the perpendicular bisector of the segment **[PQ],** so **BD** meets **PQ** at some point **X** between **P** and **Q.** By construction the points **D** and **B** lie on opposite sides of **PQ,** and this implies that the common point of the lines **BD** and **PQ** must lie between **B** and **D.** Thus we have the betweenness relationships **P$*$X$*$Q** and **B$*$X$*$D,** and these imply that **D *must lie in the interior of*** $\angle$ **ABC** (because the first betweenness relationship implies that **P** and **X** lie on the same side of the line **BC** and also that **Q** and **X** lie on the same side of the line **BA,** while the second betweenness relationship implies that **X** and **D** lie on the same sides of the lines **BA** and **BC,** and these combine to show that **X** and **D** lie in the angle interior by the definition of the latter)**.**

The proof that **|$\angle$ ABD|** = **|$\angle$ DBC|** is now just the standard argument in elementary geometry textbooks**.** The equalities **|BP|** = **|BQ|** and **|PD|** = **|QD|** and the identity **|BD|** = **|BD|** imply that $\triangle$**ABC** $\cong$ $\triangle$**DEF,** and this implies that **|$\angle$ ABD|** = **|$\angle$ DBC|.**

FOOTNOTE. The basic continuity result is Theorem 3 on page 134 (= document page 19) of the file **http://math.ucr.edu/~res/math133/geometrynotes3b.pdf.** It applies to the situation in Step 2**,** for the line **PQ** meets the circle centered at **Q** in two points**,** one of which is **P** and the other of which we shall call **Y,** and the points **P** and **Y** are respectively inside and outside the circle with center **P.** Therefore the theorem in question implies that the two circles meet in two points, one on each side of the line **PQ.**