Vector proofs of elementary results in geometry

Our goal is to prove the following classical result in Euclidean geometry:

THEOREM. *Suppose that $\angle ACB$ in the coordinate plane is inscribed in a* semicircle*; in other words, if X is the midpoint of the segment $[AB]$ then all three points $A,\ B,\ C$ are equidistant from X. Then $\angle ACB$ is a right angle.*

See the file `semicircle.pdf` for a drawing.

Proof. We shall view the points in the coordinate plane as vectors and relabel them as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{x}$. Since $\mathbf{x}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$ it follows that $\mathbf{a} - \mathbf{x} = -(\mathbf{b} - \mathbf{x})$. Let

$$r \quad = \quad |\mathbf{a} - \mathbf{x}| \quad = \quad |\mathbf{b} - \mathbf{x}| \quad = \quad |\mathbf{c} - \mathbf{x}| \ .$$

In vector language, the conclusion of the theorem is that $\mathbf{a} - \mathbf{c}$ and $\mathbf{b} - \mathbf{c}$ are perpendicular, or equivalently that

$$(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c}) \quad = \quad 0 \ .$$

Define new vectors

$$\mathbf{a}' \ = \ \mathbf{a} - \mathbf{x} \ , \quad \mathbf{b}' \ = \ \mathbf{b} - \mathbf{x} \ , \quad \mathbf{c}' \ = \ \mathbf{c} - \mathbf{x} \ .$$

It follows from the definitions that $\mathbf{a}' = -\mathbf{b}'$, and all three vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ have length r. Furthermore, we also have

$$\mathbf{a}' - \mathbf{c}' \quad = \quad \mathbf{a} - \mathbf{c} \ , \quad \mathbf{b}' - \mathbf{c}' \quad = \quad \mathbf{b} - \mathbf{c}$$

and therefore the conclusion of the theorem translates into the condition

$$(\mathbf{a}' - \mathbf{c}') \cdot (\mathbf{b}' - \mathbf{c}') \quad = \quad 0 \ .$$

Since $\mathbf{a}' = -\mathbf{b}'$, we may rewrite the expression on the left hand side as

$$(-\mathbf{b}' - \mathbf{c}') \cdot (\mathbf{b}' - \mathbf{c}') \quad = \quad -(\mathbf{b}' + \mathbf{c}') \cdot (\mathbf{b}' - \mathbf{c}') \quad = \quad -\big(\, |\mathbf{b}'|^2 - |\mathbf{c}'|^2 \,\big) \ .$$

Since $\mathbf{b}'$ and $\mathbf{c}'$ both have length r, it follows that this expression equals zero, which is what we needed to show in order to prove the theorem.