2. Greek mathematics before Euclid

(Burton, **3.1 – 3.4, 10.1**)

It is fairly easy to summarize the overall mathematical legacy that Greek civilization has left us:

> ***Their work transformed mathematics from a largely empirical collection of techniques into a subject with a coherent theme of organization, based mainly upon deductive logic.***

Logic definitely played a role in Egyptian and Babylonian mathematics, as it also did in the mathematics of other ancient civilizations, simply because problem solving involves logical reasoning. The crucial feature of Greek mathematics was that *logic was used not only to solve specific types of problems, but also to organize the subject*.

Perhaps the best known consequence of the Greek approach to mathematics is the very strong emphasis on justifying results by means of logical proofs. One important advantage of proving mathematical statements is that *it dramatically increases the reliability and accuracy of the subject, and it emphasizes the universality of the subject's conclusions*. These features have in turn greatly enhanced the usefulness of mathematics in other areas of knowledge.

Since deductive logic did not play such a central role in the mathematics of certain other major cultures, it is natural to ask how, why and when Greek mathematics became so logically structured. Various reasons for this have been advanced, and they include (i) more widespread access to education and learning, (ii) greater opportunities for independent thought and questioning so – called conventional wisdom, (iii) more emphasis on the intellectual sides of other subjects such as the arts, and (iv) more flexibility in approaches to religious beliefs. There are relationships among these factors, and at least some of them have also been present in later civilizations where mathematics flourished, including the modern world.

Logic and proof in mathematics

Before discussing historical material, it seems worthwhile to spend a little more time discussing the impact of the Greek approach to mathematics as a subject to be studied using the rules of logic. This is particularly important because many often repeated quotations about the nature of mathematics and the role of logic are either confusing or potentially misleading, even to many persons who are quite proficient mathematically.

Inductive versus deductive reasoning. Earlier civilizations appear to have reached conclusions about mathematical rules by observation and experience, a process that is known as ***inductive reasoning*** and is described in the following online reference:

http://changingminds.org/disciplines/argument/types_reasoning/induction.htm

This process still plays an important role in modern attempts to understand nature, but it has an obvious crucial weakness: A skeptic could always ask if there might be some example for which the alleged rule does not work; in particular, if one is claiming that a certain rule holds in an infinite class of cases, knowledge of its validity in finitely many cases need not yield any information on the remaining infinite number of cases.

Some very convincing examples of this sort are given by a number – theoretic question that goes back nearly two thousand years: If p is a prime, determine whether there are integers m and n such that

$$m^2 = p \cdot n^2 + 1.$$

This identity is known as ***Pell's equation***; as suggested above, this equation had been recognized much earlier by Greek mathematicians, and several Chinese and Indian mathematicians had studied it extensively during the thousand years before the work of J. Pell (1611 – 1685) in the 17th century. Here is a MacTutor reference:

http://www–groups.dcs.st–and.ac.uk/~history/HistTopics/Pell.html

For small values of p one can find solutions directly by setting $n = 1, 2, \ldots$ and checking whether $p \cdot n^2 + 1$ is a perfect square. Here are a few examples:

$$\begin{aligned} p &= 2: \ (n, m) = (2, 3) \\ p &= 3: \ (n, m) = (1, 2) \\ p &= 5: \ (n, m) = (4, 9) \\ p &= 7: \ (n, m) = (3, 8) \\ p &= 11: \ (n, m) = (3, 10) \end{aligned}$$

In fact, one can prove that this equation always has a solution where p is a prime and m and n are both positive integers. For example, see page 35 of the following reference:

http://www2.math.ou.edu/~kmartin/nti/chap5.pdf

However, even for relatively small values of p a minimal solution to this equation can involve integers which are so large that one might conclude that there are no solutions. In particular, if we take $p = 991$ and compute $991 \cdot n^2 + 1$ for a even a few million values of n, it would be easy to conclude that $991 \cdot n^2 + 1$ will never be a perfect square; in fact there are no positive integer solutions until one reaches the case

$$n = 12\,055\,735\,790\,331\,359\,447\,442\,538\,767 \sim 1.2 \times 10^{29}$$

and for this value of n the expression does yield a perfect square. An even more striking example appears in J. Rotman's undergraduate text on writing mathematical proofs: The smallest positive integer n such that

$$1000099 \cdot n^2 + 1$$

is a perfect square has 1115 digits. Here is a bibliographic reference for this book:

J. Rotman, ***Journey into mathematics. An introduction to proofs.*** *Prentice Hall, Upper Saddle River, NJ,* 1998. **ISBN:** 0–13–842360–1.

On a more elementary level, consider the following question:

If n is a positive integer, is $n^3 - n$ evenly divisible by 6?

One approach to this might be to compute the expression $n^3 - n$ for several million values of n, check that the result is always divisible by 6 in all these cases and conclude that this expression is probably divisible by 6 in all possible cases. However, in this case one can conclude that the result is always true by an argument we shall now sketch: Given two consecutive integers, we know that one is even and one is odd. Likewise, if we are given three consecutive integers, we can conclude that exactly one of them is divisible by 3. Since

$$n^3 - n = n \cdot (n+1) \cdot (n-1)$$

we see that one of the numbers n and $n+1$ must be even, and one of the three numbers $n-1$, n and $n+1$ must be divisible by 3. These divisibility properties combine to show that the entire product is divisible by $2 \cdot 3 = 6$.

One ***important and potentially confusing point*** is that the deductive method of proof called *mathematical induction* is **<u>NOT</u>** an example of *<u>inductive reasoning</u>*. Due to the extreme importance of this fact, we shall review the underlying reasons: An argument by mathematical induction starts with a sequence of propositions $\mathbf{P}_n$ such that

(1) the initial statement $\mathbf{P}_1$ is true,

(2) for all positive integers n, if $\mathbf{P}_n$ is true then so is $\mathbf{P}_{n+1}$

and concludes that ***every statement*** $\mathbf{P}_n$ ***is true.*** This principle is actually a form of proof by *reductio ad absurdum*: If some $\mathbf{P}_n$ is false then there is a least n such that $\mathbf{P}_n$ is false. Since $\mathbf{P}_1$ is true, this minimum value of n must be at least 2, and therefore $n-1$ is at least 1. Since n is the first value for which $\mathbf{P}_n$ is false, it follows that $\mathbf{P}_{n-1}$ must be true, and therefore by condition (2) it follows that $\mathbf{P}_n$ must be true. So we have shown that the latter is both true and false, which is impossible. —— What caused this contradiction? The only thing that could be responsible for the logical contradiction is the assumption that some $\mathbf{P}_m$ is false. Therefore there can be no such m, and hence each $\mathbf{P}_n$ must be true.

A geometrical example for the usefulness of deductive logic is given in the file **<u>http://math.ucr.edu/~res/math153-2020/week2/unit2/orthocenter.pdf</u>**.

<u>The accuracy of mathematical results</u>. In an earlier paragraph there was a statement that proofs greatly increase the reliability of mathematics. One frequently sees stronger assertions that proofs ensure the absolute truth of mathematics. It will be useful to examine the reasons behind these differing but closely related viewpoints.

Perhaps the easiest place to begin is with the question, "***What is a geometrical point?***" Mathematically speaking, it has no length, no width and no height. However, it is clear that no actual, observable object has these properties, for it must have measurable dimensions in order to be observed. The mathematical concept of a point is essentially ***a theoretical abstraction that turns out to be extremely useful*** for studying the spatial properties of the world in which we live. This and other considerations suggest the following way of viewing the situation: *<u>Just like other sciences, mathematics can be viewed formally as a theory about some aspects of the world in which we live.</u>* Most of these aspects involve physical quantities or objects – concepts that are also basic to other natural sciences.

If we think of mathematics as dealing only with its own abstract concepts, then one can argue that it yields universal truths. However, if we think of mathematics as providing information about the actual world of our experience, then a mathematical theory must be viewed as an idealization. As such, it is more accurate to say that the results of mathematics provide extremely reliable information and a degree of precision that is arguably unmatched in other areas of knowledge. The following quotation from Albert Einstein (1879 – 1955) summarizes this viewpoint quite well:

> As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.

Given the extent to which Einstein used very advanced mathematics in his work on theoretical physics, it should be clear that this comment did not represent a disdain for mathematics on his part, and in order to add balance and perspective we shall also include some quotations from Einstein supporting this viewpoint:

> One reason why mathematics enjoys special esteem, above all other sciences, is that its laws are absolutely certain and indisputable, while those of other sciences are to some extent debatable and in constant danger of being overthrown by newly discovered facts.
>
> But there is another reason for the high repute of mathematics: It is mathematics that offers the exact natural sciences a certain measure of security which, without mathematics, they could not attain.

This reflects the high degree of reliability that mathematics possesses due to its rigorous logical structure.

***The place of logic in mathematics*.** Despite the importance of logic and proof in mathematics as we know the subject, it is important to remember that all the formalism is a means to various ends rather than an all – encompassing end in itself. Just like other subjects, mathematical discovery follows the pattern described by Immanuel Kant (1724 – 1804) in his *Critique of Pure Reason*:

> All our knowledge begins with the senses, proceeds then to the understanding, and ends with reason.

In particular, the discovery process in mathematics uses experience and intuition to develop concepts and ideas, and the validity of the latter is determined by means of deductive logic. The following quote from Hermann Weyl (1885 – 1955) summarizes this use of logic to confirm the reliability of mathematical conclusions:

> Logic is the hygiene the mathematician practices to keep his ideas healthy and strong.

In everyday life, different standards of hygiene are appropriate for different purposes. For example, the extremely tight standards of hygiene necessary for manufacturing computer chips are clearly different from the standards that are reasonable for selling computers. The same general principle applies to logical standards for the study and uses of mathematics.

In a similar vein, although the technical language of mathematics is a very effective framework for ensuring the logical reliability of the subject, it does not always reflect the perspectives of some users of mathematics in other disciplines. Some issues along these lines are discussed in the following article:

> **G. B. Folland.** *Speaking with the natives: Reflections on mathematical communication.* Notices of the American Mathematical Society Vol. **57** (2010), pp. 1121 – 1124. Available online at **http://www.ams.org/notices/201009/rtx100901121p.pdf**.

The role of definitions in formal logic. In logical arguments it is important to be careful and consistent when stating definitions. This contrasts with everyday usage, where it is often convenient to be somewhat imprecise in one way or another. For example, if one looks up a definition in a standard dictionary and then looks up the definitions of the words used to define the original word and so forth, frequently one comes back to the original word itself, and thus from a strictly logical viewpoint the original definition essentially goes around in a circle. Often such rigorous definitions of words in mathematics have implications that are contrary to standard usage; the following quotation from the 20th century English mathematician J. E. Littlewood (1885 – 1977) illustrates this phenomenon very clearly:

> A linguist would be shocked to learn that if a set is not closed this does not mean that it is open, or again that "**E** is dense in **E**" does not mean the same thing as "**E** is dense in itself."

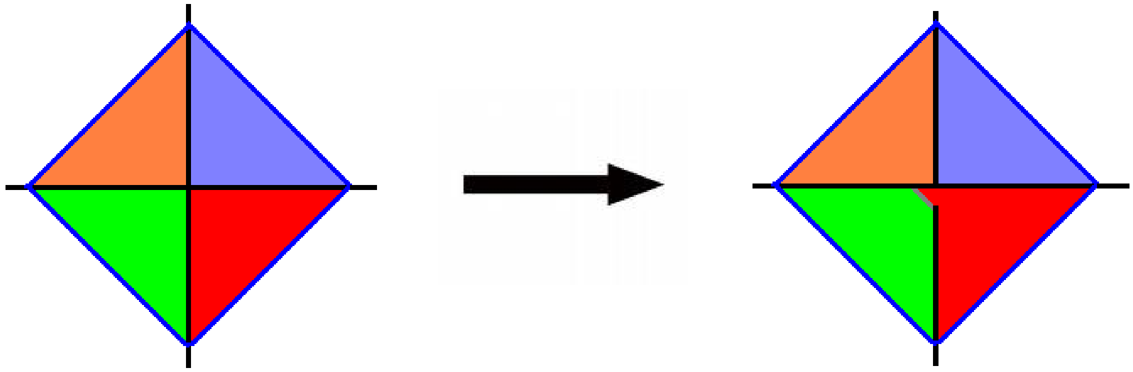
The rigidity of mathematical definitions is described very accurately in a well known quotation of C. L. Dodgson (better known as Lewis Carroll, 1832 – 1898) that appears in his classic book, ***Through the Looking Glass*** (Alice in Wonderland):

> "When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean – neither more nor less."

There will be further comments on logical definitions later in this unit.

The evolution of logical standards over time. Very few things in this world emerge instantly in a fully developed form and remain completely unchanged with the passage of time. The logical standards for mathematical proofs are no exception to this. Ancient Greek and Roman writings contain some information on the development of logic and mathematical proofs in Greek civilization. Some aspects of this process in Greek mathematics will be discussed later in these notes, and we shall also discuss the changing standards for mathematical proofs at various points in subsequent units. This continuing refinement of logical standards is frequently related to advances within mathematics itself. When mathematicians and others make new discoveries in the subject and check the logical support for these discoveries, occasionally it is apparent that existing criteria for valid proofs require a careful re – examination. Often such work is absolutely necessary to ensure the accuracy of new discoveries. In some cases such refinements of logical standards raise questions about earlier proofs, but in practice mathematicians are able to address such questions effectively with relatively minor adjustments to previous arguments.

During the past 30 to 40 years, some uses of computers in mathematical proofs have raised unprecedented concerns. Perhaps the earliest example to generate widespread attention was the original proof of the ***Four Color Theorem*** by K. Appel (1934 – 2013) and W. Haken (1928 –) in the middle of the nineteen seventies. The most intuitive formulation of this result is that four colors suffice to color a "good" map on the plane (each country consists of a single connected piece, and no boundary point lies on the boundaries of more than three countries; in particular, this eliminates phenomena like the four corners point in the U. S. where Colorado, Utah, Arizona and New Mexico all meet — as in the picture below, one can always modify boundaries very slightly to achieve this regularity condition, and this can even be done more carefully without changing the areas of any of the regions).



The original Appel – Haken proof used a computer to analyze thousands of examples of specific maps, and questions arose about the reliability of such a program. One widely held view reflects a basic principle of the Scientific Method regarding experimental results: ***In order to verify their validity, someone else should be able to reproduce the results independently.*** For computer assisted proofs, this means running another computer test on a different machine using independently written programs. In fact, tests of this sort were done for the Four Color Theorem with positive results. For further information on this result, see the discussion on pages 748 – 750 of Burton and the online article **http://en.wikipedia.org/wiki/Four_color_theorem**.

There is a more detailed but relatively informal discussion of mathematical proofs and their basic techniques in the following online document:

http://math.ucr.edu/~res/math133-2018/mathproofs.pdf

Comment on the term "Greek mathematics"

When discussing any aspect of ancient Greek culture, it is important to remember that the latter became a dominant intellectual framework over increasingly wide geographic areas as time progressed. Particularly in the Hellenistic period — which is conveniently viewed as beginning with the conquests of Alexander the Great and the founding of Alexandria — it includes contributions from many different geographical areas and nationalities in Southern and Eastern Europe, Western Asia, and Northern Africa. The justifications for the term "Greek mathematics" are the cultural foundations of the work and the fact that Greek was the usual language in which the work was written.

The time periods of Greek mathematics

The historical period for the Greek school of mathematics probably began no later than 600 B.C.E., and it continued until shortly after 400 A.D.. Not surprisingly, the methods, emphases and problems changed over this long stretch of time, and for many purposes it is useful to split the history into several distinct eras:

The ***Ionian Period*** (***c***. 600 B.C.E. – ***c***. 450 B.C.E., *so named because much of the activity took place in the Ionian Islands and the western part of Asia minor*). This period began with the establishment of mathematics as a subject in its own right and continued with numerous contributions; by the end of this period, questions about irrational numbers had become a major issue.

The ***Athenian Period*** (***c***. 450 B.C.E. – ***c***. 300 B.C.E., *so named because Athens was a center of activity during this time*). During this period the difficulties with irrationals were managed with a geometric approach to algebraic issues, and the basic framework of the subject took a more definitive form.

The ***Hellenistic Period*** (***c***. 300 B.C.E. – ***c***. 150 A.D.). This period begins with the spread of Greek culture following the conquests of Alexander the Great. During the first few centuries of this period Greek mathematics made its most important and profound contributions. Towards the end there was considerably less progress and less focus on traditional geometrical questions.

The ***Post – Hellenistic Period*** (***c***. 150 A.D. – ***c***. 400 A.D.). There was a further decline in activity, with progress only in a few directions. These included mathematical tools needed for astronomy, the organization and preservation of Greek mathematical achievements along with a numerous elaborations, and some work on number theory that had few if any precedents in Greek mathematics.

The beginnings of Greek mathematics

The early period of Greek mathematics began well over a thousand years after the period during which most surviving documents from Egyptian and Babylonian mathematics were written. However, in contrast to the primary sources we have for these cultures, our information about the earliest Greek mathematics comes from secondary sources. The writings of Proclus Diadochus (410 – 485 A.D.) are particularly useful about this period, and they make numerous references to a lost history of mathematics that was written by a student of Aristotle named Eudemus of Rhodes (350 – 290 B.C.E.) about 325 B.C.E.. There is an extensive and informative discussion of historical sources for Greek mathematics in Chapter 2 of the textbook by Hodgkin.

What, then, can we say about the beginnings of Greek mathematics? We can conclude that Greek civilization learned a great deal about Egyptian and Babylonian mathematics through direct contacts which included visits to these lands by Greek scholars during the 6^{th} century B.C.E.. We can also conclude that during this period the Greeks began to organize the subject using deductive logic — a development that has had an obvious an enormous impact both on mathematics and on other areas of human knowledge. We can also conclude that certain individuals like Thales of Miletus and Pythagoras of Samos played prominent roles in the development of the subject, both through their own achievements and through the schools of study which they led. We can also safely conclude that certain results were known during the 6^{th} century B.C.E.. However, we cannot be certain that all the biographical stories about these early mathematicians are accurate, and there is a considerable uncertainty about the proper attribution of results, quotations, and specific achievements to individuals. Our discussions of the earliest Greek mathematicians should be viewed in this light. In particular, it is probably better to view the progress during this period very reliably as the legacy of a culture and less reliably as the legacy of specific individuals who became legendary figures.

Thales of Miletus ($c.$ 624 – 548 B.C.E.) is the first individual to be credited with specific mathematical discoveries and contributions (Miletus was near the south end of the west coast of Asia Minor). He was one of the renowned Seven Sages in ancient Greek tradition and Plato's ***Protagoras*** (the others on his list were Pittacus of Mytilene, Bias of Priene, Solon of Athens, Cleobulus of Lindus, Myson of Chen, and Chilon of Sparta; see **http://www.infoplease.com/ce6/history/A0844573.html**).

Several basic theorems in geometry are attributed to Thales. The Vertical Angle Theorem (page 7 of **http://math.ucr.edu/~res/math133-2018/geometrynotes02b.f13.pdf**) is one example, and another is that an angle inscribed in a semicircle is a right angle (see **http://math.ucr.edu/~res/math153-2019/circleright.pdf**; for an alternative proof see **http://math.ucr.edu/~res/math153-2019/circleright2.pdf**). Regardless of whether the classical attributions of various proofs to Thales are correct (see page 87 of Burton), it seems clear that he contributed to the organization of mathematical knowledge on logical rather than on empirical grounds. Thales is also credited with using basic ideas about similar triangles to make indirect measurements in situations where direct measurements were difficult or impossible. Two examples, mentioned on pages 87 – 89 of Burton, are measuring the height of the Great Pyramid by means of shadows and finding the distance from a boat to the shoreline.

We have already mentioned that Babylonian mathematicians were acquainted with the formula we know as the Pythagorean Theorem, and although it seems clear that the Pythagorean school knew the result quite well, there is no firm evidence whether or not they actually found a proof of this result; in fact, the popular story about sacrificing an animal in honor of the discovery is totally inconsistent with Pythagorean philosophy, and as such it has little credibility. However, Pythagoras of Samos ($c.$ 580 – 500 B.C.E.) and his school had a major impact on the development of mathematics that we shall now discuss (Samos is an island off the west coast of Asia Minor towards the south; since Pythagoras taught and lived in southern Italy near Crotona, which is a city in Calabria, he is also known as Pythagoras of Crotona). Given that the Pythagorean School was extremely reclusive, it is particularly difficult to make any attributions of their work to specific individuals.

One major contribution of the Pythagorean School was their adoption of mathematics as a fundamental area of human knowledge. In fact, classical Greek writings indicate that mathematics was their foundation for an all – encompassing perspective of the world, including politics, religion, and philosophy. Their program of study consisted of number theory, music, geometry and astronomy. Some examples of their teachings in these areas are described on page 93 of Burton. Here are a few online references for further information regarding their theory of musical harmonics; the first two somewhat detailed but still fairly accessible (with interactive audio examples), while the third goes into a great deal of detail on the subject and compares the Pythagorean musical scale with later versions.

http://en.wikipedia.org/wiki/Pythagorean_tuning

http://members.cox.net/mathmistakes/music.htm

http://www.friesian.com/music.htm

The Pythagoreans were intensely interested in properties of numbers, and many of their speculations on philosophical properties of numbers reflect a strong tendency towards mysticism. However, this fascination with numbers led to the discovery of many interesting and important relationships, including some that are still sources of unsolved problems.

Although there are questions whether the Pythagorean school actually gave a proof for the result we call the Pythagorean Theorem, it is clear that their studies of this result yielded some important advances. Certainly the most far – reaching was the discovery that the square root of 2 is not rational (however, some colorful stories about this discovery are highly questionable); see pages 109 – 110 of Burton for a proof of this result. Subsequently others recognized additional examples of irrational square roots, and Theaetetus ($c.$ 417 – 369 B.C.E.) proved the definitive result: *The square root of a positive integer* n *is never rational unless* n *is a perfect square*.

The existence of such irrational numbers had an enormous impact on the development of ancient Greek mathematics, and apparently it is largely responsible for the Greek emphasis on geometrical rather than algebraic methods. In particular, Greek mathematics made a clear distinction between "numbers" which were ratios of positive integers and geometrically measurable magnitudes that included quantities like $\mathrm{sqrt}(2)$. The relation between these two concepts continued to be a source of difficulties for Greek mathematics until much later work by Eudoxus of Cnidus (408 – 355 B.C.E.) that we shall discuss in the next unit on Euclid's ***Elements*** (Cnidus, now Knidos, is near the extreme southwest tip of Asia minor).

Burton discusses several other aspects of numbers that the Pythagoreans reportedly studied. Two specific contributions involve the concepts of *perfect numbers* and *amicable pairs*; an Addendum to this unit (2E) discusses still other types of numbers of interest to the Pythagoreans (namely, *polygonal numbers*).

Perfect numbers. The definitions require a few preliminaries: Given two positive integers b and c, we say that b ***evenly divides*** c provided c is an integral multiple of b, and that b is a ***proper divisor*** of c if b is strictly less than c. In these terms, a positive integer n is said to be a ***perfect number*** if it is equal to the sum of its proper divisors. The first two perfect numbers are $6 = 1 + 2 + 3$ and $28 = 1 + 2 + 4 + 7 + 14$. Euclid's ***Elements*** contains the following general method for constructing perfect numbers: *If* $2^p - 1$ *is prime, then* $2^{p-1} \cdot (2^p - 1)$ *is a perfect number.*

A proof of this result appears on pages 504 – 505 of Burton, and as noted there a result of L. Euler (1707 – 1783) shows that every even perfect number has this form (for a proof see **http://primes.utm.edu/notes/proofs/EvenPerfect.html**). It is not known whether odd perfect numbers exist; results to date show that there are no odd perfect numbers less than 10^{15000}. Here is a reference:

> **P. Ochem and M. Rao,** Odd Perfect Numbers Are Greater than 10^{15000}. *Math. Comput.* **81** (2012), 1869 – 1877.

The description of even perfect numbers leads naturally to the following question: *For which integers* p *is* $2^p - 1$ *a prime number*? Simple algebra shows this can only happen if p is prime, but in 1536 Hudalricus Regius showed that the integer $2^{11} - 1 =$

2047 is equal to $23 \cdot 89$, and in fact there are many primes p for which $2^p - 1$ is not prime. A prime number of this form is called a ***Mersenne prime*** in recognition of the influence which M. Mersenne (1588 – 1648) had on the study of such numbers. Section 10.1 of Burton contains a detailed discussion on the important role Mersenne played in furthering communication among 17th century mathematicians as well as additional information on perfect numbers and Mersenne primes.

During the past 65 years, computer calculations have greatly expanded the list of known Mersenne primes from **17** to **51**, with the most recent addition to the list announced at the end of 2018. The largest known Mersenne prime has **24,862,048** digits, and further information appears in the following basic source:

http://www.mersenne.org/prime.htm

One outstanding open question is whether there are finitely or infinitely many Mersenne primes (see **http://en.wikipedia.org/wiki/Mersenne_prime** for further information on this and a few other related topics).

Amicable pairs. A pair of positive integers is said to be an ***amicable pair*** if each is equal to the sum of the proper divisors of the other. Although it might not be obvious that such pairs exist, the Pythagoreans reportedly knew that **220** and **284** form an amicable pair. Further material on this topic appears on page 510 of Burton. Fairly recent summaries of known results (up to 2007) appear on the following sites:

http://mathworld.wolfram.com/AmicablePair.html

http://amicable.homepage.dk/knwnc2.htm

***Final notes*:** **(1)** There is evidence that the amicable pair **{17296, 18416}** discovered by P. Fermat in the 17th century had been known to Thābit ibn Qurra (Al-Ṣābi' Thābit ibn Qurra al-Ḥarrānī, 826 – 901), who gave a general criterion for recognizing amicable pairs that is stated on page 510 of Burton. As noted on that page, we do not have a characterization of amicable pairs comparable to the simple criterion for even perfect numbers due to Euclid and Euler. Several basic results on amicable pairs are stated and proved in the following online documents:

http://www.maa.org/editorial/euler/How%20Euler%20Did%20It%2025%20amicable%20numbers.pdf

http://mathdl.maa.org/images/upload_library/22/Evans/pp.05-07.pdf

(2) Most history of mathematics books note that in 1866 a 16 – year old student named B. Nicolò I. Paganini discovered the previously unknown amicable pair **{1184, 1210}.** We should emphasize that he was **NOT** the famous 19th century violinist and composer with the very similar name of Niccolò Paganini (1782 – 1840).

The "Heroic Age" and its aftermath

This period covers all of the 5th century B.C.E. and most of the following century. The dominant influences during the period were various groups of scholars, particularly in the Elean, Sophist and Platonic schools. Since a great deal of misinformation about these schools is presented as factual in popularized writings on culture and philosophy, here are some background references which are relatively concise but reasonably accurate:

http://en.wikipedia.org/wiki/Eleatics

http://en.wikipedia.org/wiki/Sophism

http://www.utm.edu/research/iep/s/sophists.htm

http://en.wikipedia.org/wiki/Platonism

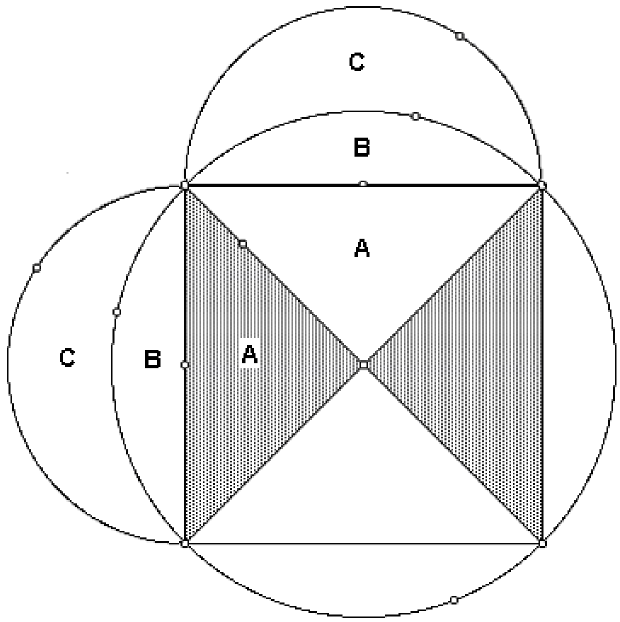
http://www.iep.utm.edu/greekphi/

Although the Pythagorean school was not as prominent during this period as it was in the preceding century, it continued to exist. One of its last members was Archytas of Tarentum (428 – 347 B.C.E), who was a contemporary of Plato; Eudoxus of Cnidus (whom we shall discuss later) was his mathematical student (Tarentum, now Tarento, is a city on the coast of the southern Italian province of Apulia).

Probably the best known member of the Elean school was Zeno of Elea (*c.* 490 – 430 B.C.E.), whose challenging paradoxes about moving objects have attracted attention and generated controversy ever since they were first stated (Elea, now called Velia, is in the Campania province of southern Italy). These problems illustrate the difficulties that arise if one is too casual about mixing discrete and continuous physical models, and it seems likely that the paradoxes were part of a continuing discussion of empirical versus abstract logical observation; there is a detailed discussion of these problems in **https://plato.stanford.edu/entries/paradox-zeno/**. One paradox ("Achilles and the tortoise") is described on page 103 of Burton; it purports to show that the faster Achilles will never overtake the slower tortoise (a few illustrations remarks also appear in the file **http://math.ucr.edu/math153-2020/week2/unit2/zeno.pdf**). Clearly Zeno's conclusion is physically absurd, but finding the flaw in the argument requires ideas unknown to the ancient Greeks. All of Zeno's known paradoxes involve a sequence of time intervals such that each is half the preceding one, and in modern language they implicitly assume that the sum of all these terms diverges. Of course, today we know that the sum of the terms is just twice the initial time interval, but the study of convergent infinite series began about 1500 years after the paradoxes were first stated (specifically, during the 13th century in China and the 14th century in Europe).

The work of Hippocrates of Chios (470 – 410 B. C. E., **NOT** the celebrated physician Hippocrates of Kos, 460 – 377 B. C. E.) illustrates the evolution of Greek mathematics between the time of the Pythagoreans and the later eras of Plato and Euclid (Chios and Kos are islands off the central west and southwest coast of Asia Minor; Chios is also known for an unusual rocket war celebration which takes place before Easter – see the video **https://www.youtube.com/watch?time_continue=10&v=_PijfPZx88I**). Hippocrates took important steps towards the systematic development of geometry from an axiomatic viewpoint, he was one of the first persons credited with using the technique of proof by contradiction (or ***reductio ad absurdum***), and he wrote the first text on the elements of geometry more than a century before Euclid's ***Elements***.

Hippocrates of Chios is also known for his results on computing the areas of certain geometric figures called ***lunes***; these are plane regions bounded by a pair of circular arcs with different radii. In the figure below, the (congruent) regions marked with the letter **C** are lunes determined by two circles such that the diameter of one is equal to **sqrt(2)** times the diameter of the other.



Hippocrates' determination of the area **C** bounded by either lune proceeds as follows (compare pages 122 – 124 of Burton): He knew that the area of a semicircular region was proportional to the square of the radius. Therefore the area of the larger semicircle, which is $2(\mathbf{A} + \mathbf{B})$, must be twice that of the smaller semicircle, which is $\mathbf{B} + \mathbf{C}$; therefore we have $2(\mathbf{A} + \mathbf{B}) = 2(\mathbf{B} + \mathbf{C})$. Simple algebra now tells us that $\mathbf{A} = \mathbf{C}$. Given the irregular shape of the lune, it is not immediately obvious that its area should be given by such a simple expression, but the argument shows the lune's area can be expressed very simply. Of course one can use integral calculus to express the area of the lune as a definite integral, and computing the latter turns out to be an interesting and somewhat challenging exercise. Hippocrates' investigations also yielded simple formulas for the areas of certain other types of lunes; additional information on these and subsequent results are summarized in the following online document:

http://math.ucr.edu/~res/math153-2019/history02d.pdf

Hippocrates' proof provides some interesting insights into the level of Greek geometry at the time. First, it indicates that Greek mathematics had attained a fairly good level of proficiency in manipulating geometric quantities during this period. Second, it illustrates the usefulness of deductive reasoning to discover information that is not intuitively obvious and not likely to be discovered by empirical means. Third, the argument does not really need the explicit computation of the area of the region enclosed by a circle of radius r (the usual $\mathbf{AREA} = \pi r^2$), which probably was not known at the time, but instead it uses just a weaker proportionality statement; as such, the proof indicates an ability to find ways of working around obstacles to reach the objective. However, Hippocrates' result is also noteworthy because of its relation to the three "impossible" construction problems of antiquity, which had already attracted considerable attention in Greek mathematics before Hippocrates' work. Informal statements of the three problems are well known, but for our purposes it is worthwhile to state them formally.

The three "impossible" classical construction problems. Using only an **unmarked** straightedge (which is **NOT** a modern ruler with marked off distances**!**) and a **collapsible** compass, carry out the following constructions:

1. Trisect an arbitrary angle.
2. Find a square whose area is equal to that of a given circle.
3. Find a cube whose volume is twice that of a given cube.

Each of these is a natural sequel to known constructions from Greek geometry. In particular, Greek geometers knew how to bisect an arbitrary angle with straightedge and compass (see **http://math.ucr/edu/~res/math153-2019/bisection.pdf**), how to find a square whose area is twice that of a given square, and how to find a square whose area is equal to that of a given equilateral triangle. We shall say more about the three construction problems later (see also pages 124 – 127 of Burton), and an explanation of why these problems are impossible to solve is given in a supplement to these notes; for now we simply note that Hippocrates' results on lunes were possibly byproducts of efforts to solve the second problem.

The Sophists in ancient Greek civilization were particularly interested in these three construction problems. Competing schools of thought at the time were strongly critical of the Sophists for several reasons, but for our purposes these controversies can be ignored and we shall concentrate on mathematical achievements. One of the best known Sophists was Hippias of Elis (460 – 400 B. C. E.), who made an early and significant contribution to these problems; specifically, the introduction of curves other than straight lines and circles (Elis, which corresponds to the present day Greek prefecture Ilias, was in southern Greece on the Peloponnesus peninsula). This curve, the ***quadratrix*** or ***trisectrix of Hippias***, is discussed from the classical viewpoint on pages 130 – 134 of Burton. Its equation in polar coordinates is

$$r \;=\; \theta/\sin\theta$$

and its equation in Cartesian coordinates is

$$x \;=\; y\cdot\cot y.$$

The discussion on pages 132 – 134 of Burton indicates how this curve can be used to trisect angles and find a square whose area equals that of a given circle. Further information on this curve may be found at the online sites listed below. The second site has a link to an animated graphic tracing the motion of an object along the curve which corresponds to the classical Greek definition.

http://www-groups.dcs.st-and.ac.uk/~history/Curves/Quadratrix.html

http://xahlee.info/SpecialPlaneCurves_dir/QuadratrixOfHippias_dir/quadratrixOfHippias.html

http://oinou.nl/quadratrix-van-hippias/

Classical writers assert that Hippias used the curve to trisect angles and the application to squaring circles was completed later by Dinostratus (390 – 320 B.C.E.), but others have claimed that this was also known to Hippias. In any case, beginning in the Heroic age, several Greek mathematicians developed many different curves that could be used to solve the three classical construction problems, and some of these curves have proven to be extremely important in mathematics and its applications to physics; a few examples are described in the online references given above. We shall discuss some of these curves in the unit on Greek mathematics after Euclid.

Plato's mathematical legacy

Finally, we come to the Platonic school. In any discussion of ancient Greek knowledge and thought, it is nearly impossible to avoid mentioning Socrates (*c.* 469 – 399 B.C.E.), Plato (428/427 – 348/347 B.C.E.) and Aristotle (384 – 322 B.C.E.). Before discussing the mathematical aspects of their contributions, we shall give a link to an online history (it is a little clumsy to use, but it contains a great deal of information).

http://www.perseus.tufts.edu/hopper/text?doc=Perseus:text:1999.04.0009

Although Socrates was uninterested, and perhaps even negative, about mathematics, Plato and his students had a major impact on the subject. Plato himself was not a mathematician, but his views had a major influence on the subject in several ways:

(1) He insisted on a more rigorous logical framework for doing mathematics, including carefully formulated definitions, axioms, postulates and strict sequential development.

(2) His idea of viewing mathematics formally as an idealized model for reality became a standard for future thought, up to and including the present.

(3) His insistence on constructions by straightedge and compass became an ideal for much future work on the subject even though there was also much work on solving the three "impossible" classical construction problems by other methods.

(4) His emphasis on some aspects of solid geometry, particularly on the five regular Platonic solids, spurred further interest in this area.

We shall say more about the five regular Platonic solids later, but for the time being we describe them briefly. The most basic are the ***cube*** and the ***tetrahedron***; the latter is a triangular pyramid whose base and sides are all equilateral triangles. The centers of the six faces of a cube are the vertices of another regular solid called the ***octahedron***, which consists of eight equilateral triangles, with each vertex lying on exactly four of them (one can also think of this as a pair of pyramids with square bases which are glued together along the square bases). There also is the ***dodecahedron***, which is a configuration of regular pentagons having twelve faces where each vertex lies on three of the faces, and finally there is the ***icosahedron***, which has twenty equilateral triangles and five meeting at each vertex. Illustrations of all five solids are given in the online file **http://math.ucr.edu/~res/math153-2019/history03d.pdf**.

The specific mathematical contributions of the Platonic school are due to some of Plato's students who pursued mathematics and also to some students of these students. We have already mentioned some contributions of Theaetetus to the study of irrational numbers, and in the section on Euclid we shall discuss the contributions of Eudoxus of Cnidus. The latter is also known for his approach to computing the area bounded by a circle by using inscribed and circumscribed regular polygons; specifically, the idea was that one obtained increasingly better approximations by taking polygons with greater numbers of sides, and in the limit the areas enclosed by these polygons became the area bounded by the circle. This ***method of exhaustion*** very clearly anticipated the methods of integral calculus for finding areas using approximations by more manageable figures.

Aristotle's main contribution was of a somewhat different nature; although he was not a mathematician, his work on logic further refined the role of that subject as a formal setting for mathematics.

Systems of linear equations

Although Greek mathematics did not reach the level of algebraic proficiency seen in certain other cultures, one noteworthy advance was a result on solving systems of linear equations due to Thymaridas of Paros (*c.* 400 B.C.E. – *c.* 350 B.C.E.). The system, known as the ***bloom of Thymaridas***, is given by

$$
\begin{aligned}
x + x_1 + x_2 + \ldots + x_{n-1} &= s \\
x + x_1 &= m_1 \\
x + x_2 &= m_2 \\
&\ldots \\
x + x_{n-1} &= m_{n-1}
\end{aligned}
$$

and the solution is $x = [(m_1 + m_2 + \ldots + m_{n-1}) - s]/(n-2)$.

Addenda to this unit

There are seven separate items. The first (**2A**) describes a method for computing the area of Hippocrates' lune using integral calculus, the second (**2B**) discusses the reasons why the three classical construction problems cannot be solved by means of an unmarked straightedge and a compass, the third and fourth (**2C, 2Ca**) discuss a naïve but incorrect attempt to trisect angles, the fifth (**2D**) contains still more information about areas of lunes, the sixth (**2E**) gives additional information on polygonal numbers, and the seventh (**2F**) is about interactions between mathematics and subjects that are either mystical or now considered to be pseudoscientific (such as astrology, numerology and the occult side of alchemy). Further information on the topics in (**2C**) can also be found at the following site:

http://www.jimloy.com/geometry/trisect.htm

In particular, the portion called

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specifically discusses further elaborations of the construction in (**2C**). The entire Jim Loy mathematics site

http://www.jimloy.com/math/math.htm

also treats many other topics that are relevant to high school and lower level college mathematics in an accessible and mathematically correct manner, and it is highly recommended.

Not surprisingly, there is a long history of failed attempts to solve the three classical construction problems described in this unit (and the final chapters are probably yet to be written!). The following book describes a few noteworthy examples:

U. Dudley, ***The Trisectors*** (2nd Ed.). *Mathematical Association of America*, *Washington DC*, 1996. **ISBN:** 0–883–85514–3.

In the author's words, "Hardly any mathematical training is necessary to read this book. There is a little trigonometry here and there **…** The worst victim of mathematics anxiety can read this book with profit."

Finally, two more documents should also be mentioned. There are two maps in each of the files

http://math.ucr.edu/~res/math153-2020/week2/unit2/historical-maps1.pdf

http://math.ucr.edu/~res/math153-2020/week2/unit2/historical-maps1a.pdf

showing the locations for many cities in Greece and Italy that are mentioned in this unit.