2.C.a. More on the purported "trisection"

Here is a conceptual and general proof for the inequality

$$\frac{1}{3}\tan x \quad > \quad \tan\frac{x}{3}$$

which is valid for $0 < x < \frac{\pi}{2}$ and was mentioned in `history02.pdf`. We shall do this using basic results from first year calculus.

We know that

$$\frac{1}{3}\tan 0 \quad = \quad 0 \quad = \quad \tan\frac{0}{3}$$

so by the Mean Value Theorem it is enough to veriify that

$$\frac{d}{dx}\left(\frac{1}{3}\tan x \; - \; \tan\frac{x}{3}\right) \quad > \quad 0$$

for $0 < x < \frac{\pi}{2}$. If we write out this derivative explicitly using the Chain Rule, we see that it is equal to

$$\frac{1}{3}\sec^2 x \; - \; \frac{1}{3}\sec^2\frac{x}{3}$$

so it suffices to check that this expression is positive for the given values of x. Now $\sec x = 1/\cos x$, and since $\cos x$ is a strictly decreasing function between 0 and $\frac{\pi}{2}$ it follows that $\sec x$ and $\sec^2 x$ are strictly increasing for $0 \leq x < \frac{\pi}{2}$. This implies that

$$\frac{1}{3}\sec^2 x \; - \; \frac{1}{3}\sec^2\frac{x}{3} \quad > \quad 0$$

for $0 < x < \frac{\pi}{2}$, and as previously noted this is exactly what we needed in order to prove the inequality in the first paragraph.■