A VECTOR PROOF OF THE VERTICAL ANGLE THEOREM

Throughout this course there are many examples of theorems which are much easier to verify with analytic methods — generally formulated in terms of vector geometry — than with the synthetic methods of classical Greek geometry. However, there is also a significant number of theorems for which the synthetic arguments are better, and this is one reason for employing a combined synthetic-analytic approach to geometry. To illustrate this point, we shall give a proof of the Vertical Angle Theorem (Theorem II.3.7 in the notes) using vector geometry.

RECALL: *The hypothesis of the Vertical Angle Theorem is that we have four distinct points* A, B, C, D *on two lines such that* $A * X * C$ *and* $B * X * D$ *hold, and the objective is to prove that* $|\angle AXB| = |\angle CXD|$.

Proof using vector geometry. By Theorem II.2.3 the betweenness conditions imply that

$$C \;=\; X \;+\; t(A - X)\;, \qquad D \;=\; X \;+\; u(B - X)$$

where t and u are both negative. For the sake of notational simplicity write $\mathbf{v} = V - X$, where V is A, B, C or D. Then we have $\mathbf{c} = t\,\mathbf{a}$ and $\mathbf{d} = u\,\mathbf{d}$, and hence the cosine of $\angle CXD$ is given by

$$\cos\angle CXD \;=\; \frac{\langle \mathbf{c}, \mathbf{d}\rangle}{|\mathbf{c}|\,|\mathbf{d}|} \;=\; \frac{\langle t\,\mathbf{a}, u\,\mathbf{b}\rangle}{|t\,\mathbf{a}|\,|u\,\mathbf{b}|} \;=$$

$$\frac{tu\,\langle \mathbf{a}, \mathbf{b}\rangle}{|t|\,|u|\,|\mathbf{a}|\,|\mathbf{b}|} \;=\; \mathrm{sgn}(t)\,\mathrm{sgn}(u)\cdot\frac{\langle \mathbf{a}, \mathbf{b}\rangle}{|\mathbf{a}|\,|\mathbf{b}|}$$

where the function $\mathrm{sgn}(y)$ is $+1$, 0 or -1 depending upon whether y is positive, negative or zero. Since both t and u are negative we have $\mathrm{sgn}(t) = \mathrm{sgn}(u) = -1$, so that the product of these two values is $+1$. Therefore the right hand expression in the chain of equations is equal to

$$\frac{\langle \mathbf{a}, \mathbf{b}\rangle}{|\mathbf{a}|\,|\mathbf{b}|} \;=\; \cos\angle AXB$$

and therefore we have $\cos\angle CXD = \cos\angle AXB$. Since the cosine function is 1–1 from the open interval $(0^\circ, 180^\circ)$ to $(-1, 1)$, it follows that $|\angle CXD| = |\angle AXB|$.■