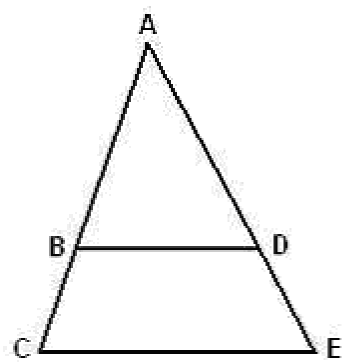
3.B. Geometric proportions and the Condition of Eudoxus

One important motivation for the Condition of Eudoxus was to consider geometric ratios involving ***incommensurable quantities***; in modern language, these are lengths **|WX|** and **|YZ|** such that the quotient **|WX|** / **|YZ|** is ***not*** a rational number. A basic problem of this nature is depicted in the figure below:



In this picture the lines **BD** and **CE** are assumed to be parallel, and one wants to prove that

$$|AB|/|AC| \quad = \quad |AD|/|AE|.$$

If the left hand side is a rational number p/q, then standard manipulations of ratios show that

$$|AB|/p \quad = \quad |AC|/q$$

and ideas discussed in the proof of the ***Notebook Paper Theorem*** imply that

$$|AD|/p \quad = \quad |AE|/q$$

which then quickly yields $|AD|/|AE| = p/q = |AB|/|AC|$. Of course, this argument breaks down completely for ratios $|AB|/|AC|$ which are equal to irrational numbers like $\mathrm{sqrt}(2)$, and we need the Condition of Eudoxus to handle such cases.

Here is a formal statement of Eudoxus' criterion for two ratios to be equal:

Two ratios of (positive real) numbers a/b *and* c/d *are equal if and only if for each pair of positive integers* m *and* n *we have the following*:

$$ma < nb \quad \textit{implies} \quad mc < nd$$

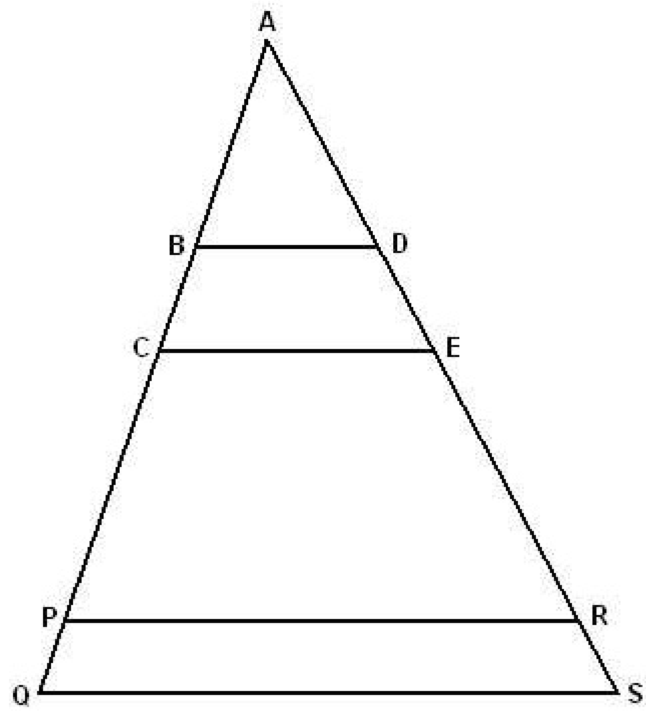
$$ma > nb \quad \textit{implies} \quad mc > nd$$

The derivation of this criterion is based upon a fundamentally important **rational density property** of the real numbers:

If we are given real numbers x *and* y *such that* $x < y$, *then there is a rational number* r *such that* $x < r < y$.

Further details about this implication are contained in the first supplement to this unit (see **http://www.math.ucr.edu/~res/math153-2019/history03a.pdf**).

APPLICATION OF THE CONDITION OF EUDOXUS TO PROPORTIONALITY QUESTIONS. Suppose now that we have triangles $\triangle$**ABD** and $\triangle$**ACE** as in the figure below, where **BD** is parallel to **CE;** as in the figure we assume that the rays **[AB** and **[AC** are the same and likewise that the rays **[AD** and **[AE** are the same. Let a = **|AB|**, b = **|AC|**, c = **|AD|** and d = **|AE|**. We want to use the Condition of Eudoxus to conclude that $a/b = c/d$.



Suppose first that m and n are positive integers such that $ma < nb$. We want to show that $mc < nd$. We can find points **P** and **Q** on the ray **[AB** = **[AC** such that **|AP|** = ma and **|AQ|** = nb. Since $ma < nb$, it follows that **P** is between **A** and **Q.** One can then find unique parallel lines to **BD** and **CE** through **P** and **Q.** These two lines will meet the line **AD** = **AE** in two points **R** and **S.** A proper formulation of concepts like betweenness, the two sides of a line, and so on will imply that **S** and **R** also lie on the ray **[AD** = **[AE** and that **R** is between **A** and **S.**

The proportionality results in the commensurable case now imply that

|AR|/|AD| = m = **|AP|/|AB|** and

|AS|/|AE| = n = **|AQ|/|AC|.**

Therefore **|AR|** = mc and **|AS|** = nd also hold. By observations in the previous paragraph we know that **|AR|** < **|AS|**, and thus we may use the preceding sentences to rewrite this as $mc < nd$. To summarize, we have now shown that $ma < nb$ implies $mc < nd$.

If we have $ma > nb$, then we may proceed similarly. The argument is basically the same except that **Q** will be between **A** and **P**, and this will in turn imply that **S** is between **A** and **R**. Following the same line of reasoning in this case, one concludes that $ma > nb$ implies $mc > nd$. Therefore we have established both parts of the Condition of Eudoxus, and consequently we have shown that $a/b = c/d$; by definition of the numbers a, b, c, d in this equation, the desired proportionality equation **|AB|/|AC|** = **|AD|/|AE|** is an immediate consequence.