Congruence and Euclidean similarity

In **http://math.ucr.edu/~res/math153-2020/week4/unit04/history04Z.pdf** we noted that there is an Angle – Angle – Angle congruence theorem in spherical geometry. This may seem contrary to intuition, but we tried to explain this is in two steps.

1. If two Euclidean triangles T_1 and T_2 satisfy Angle — Angle — Angle congruence **and they have the same area**, then they are congruent.

2. Within a fixed sphere, the area of a spherical triangle is a fixed multiple of the excess of its angle sum over 180 degrees. In general the area only depends upon this excess and the radius of the sphere. From this perspective the spherical congruence theorem has an analog in Euclidean geometry.

Although the first result rarely if ever appears in an elementary geometry course or text, it is fairly easy to verify. The Angle — Angle — Angle assumption in Euclidean geometry implies that the two triangles under consideration are similar. Let r denote the radius of similitude, so that the lengths of the triangles in the second triangle are r times those of the first. We can then use Heron's area formula to conclude that the area A_2 of the second formula is r^2 times the area A_1 of the first. We can now combine this and proportionality of the sides with the usual area formula $A = \frac{1}{2}bh$ to conclude that the altitudes satisfy $h_2 = rh_1$.

If the areas are equal then the preceding equations imply that the ratio of similitude r must be equal to 1. But this means that the ratio of the lengths of the sides in T_2 to those in T_1 must also be equal to 1. This translates into concluding the triangles satisfy the Side — Side — Side hypothesis, which means that the triangles are in fact congruent, which is what we wanted to prove.