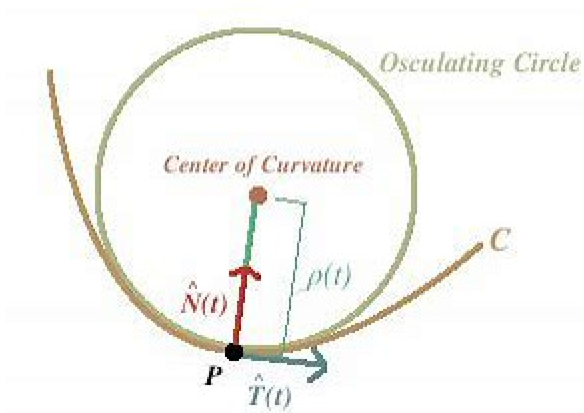
The evolute of a plane curve

This concept is implicit in the work of Apollonius on conics, but the general study began about 1800 years later in work of C. Huygens (1629 – 1695). The ***evolute*** of a plane curve is a new curve constructed out of an given one. Its description depends upon the notion of ***curvature*** for a plane curve, so we shall describe this first.

Suppose that $\mathbf{p}(t) = (x(t), y(t))$ is a parametrized curve in the coordinate plane, and assume the coordinate functions have as many derivatives as needed for the discussion to be meaningful. Then the velocity vector $\mathbf{v}(t) = (x'(t),\ y'(t))$ is obtained by differentiating the coordinate functions. Let's assume that the curve is never instantaneously at rest; mathematically this corresponds to assuming that the velocity vector is never zero. In this case there is a change of variables $s = s(t)$ such that the velocity vector $\mathbf{v}(s)$ for the reparametrized curve $\mathbf{p}(s)$ always has unit length. The ***curvature*** $\kappa(s)$ is then defined to be $1/|\mathbf{v}'(s)|$ provided that the denominator is not zero. It turns out that $\mathbf{v}'(s)$ and $\mathbf{v}(s)$ are perpendicular to each other if the latter holds, and in this case we define the principal unit normal vector $\mathbf{n}(s)$ to be the unit vector pointing in the same direction as $\mathbf{v}'(s)$.

If the curve traces out a circle of radius $r \neq 0$, then the curvature at every point turns out to be $1/r$. In a very precise sense the curvature measures how much the curve is bending at a given point; higher curvature means the curve is bending more. Specifically, if we define the **osculating circle** to be the circle which passes through the point $\mathbf{p}(s)$ and whose center is given by $\mathbf{p}(s) - \kappa(s)^{-1}\mathbf{n}(s)$, then the osculating circle can be viewed as the best circular approximation to the original curve at parameter value s.



(***Source:*** **http://mathonline.wikidot.com/the-osculating-circle-at-a-point-on-a-curve**)

We can now define the evolute $\mathbf{Ep}(s)$ of $\mathbf{p}(s)$ such that $\mathbf{Ep}(s)$ is the center of the osculating circle for the original curve at the point $\mathbf{p}(s)$.