4.C. Continued fraction expansions

Given two positive numbers a and b, we define the simple reciprocal sum expression $\mathbf{RS}(a,b)$ by the formula

$$\mathbf{RS}(a,b) \quad = \quad \frac{1}{a+b} \ .$$

The theory of continued fraction expansions depends upon the following simple observation:

THEOREM. *Let x be a real number such that $0 < x < 1$. Then $x = \mathbf{RS}(a,b)$ where a is a positive integer and $0 \leq b < 1$. If x is rational then so is b.*

The derivation of this result is simple, for we know that $(1/x) > 1$, and hence

$$\frac{1}{x} \quad = \quad a \ + \ b$$

where a is a positive integer and $0 \leq b \leq 1$; note that if x is rational then so is b.

If $x = \mathbf{RS}(a,b)$ as above and $b > 0$, then we can iterate this process, for then $b = \mathbf{RS}(a',b')$ where a' is a positive integer and $0 \leq b' < 1$, so that

$$x \quad = \quad \mathbf{RS}\big(a, \mathbf{RS}(a',b')\big) \quad = \quad \frac{1}{a + \frac{1}{a'+b'}} \ .$$

Once again, if $b' > 0$ we can apply the same consturction to b'.

We shall restrict attention here to *rational* numbers x such that $0 < x < 1$; the irrational case (which is important mathematically) is discussed in the reference for continued fractions listed in `history04Z.pdf`.

PROPOSITION. *Let x_0 be a rational number such that $0 < x_0 < 1$, so that $x_0 = \mathbf{RS}(n_1, x_1)$, where n_1 is a positive integer and $0 \leq x_1 < 1$ is rational. If we are given a pair of sequences $\{n_i\}$ and $\{x_i\}$ for $i \leq k$ such that each n_i is a positive integer and $0 < x_i < 1$, define n_{k+1} and x_{k+1} such that $x_k = \mathbf{RS}(n_{k+1}, x_{k+1})$ as before, and terminate the sequence at this point if and only if $x_{k+1} = 0$. Then there is some positive integer m such that the sequence terminates at step m; in other words, eventually one has $x_m = 0$.*

This proposition implies that every rational number x between 0 and 1 has a finite continued fraction expansion. Specifically, given x_0 satisfying $0 < x_0 < 1$ consider the sequences of numbers $\{x_k\}$, $\{n_k\}$, $\{y_k\}$ defined recursively by the conditions

(*i*) $n_0 = 0$ and $y_0 = 1/x_0$,

(*ii*) if y_k is defined with $y_k > 1$ and $y_k = \mathbf{RS}(n_{k+1}, x_{k+1})$ as in the first theorem (so that n_{k+1} is a positive integer and $0 \leq x_{k+1} < 1$), then $y_{k+1} = 1/x_{k+1}$ if $x_{k+1} > 0$, and no further terms in any of the sequences are defined if $x_{k+1} = 0$.

Then the conclusion is that for some $m \geq 1$ we get $x_{m+1} = 0$, and for $0 \leq j \leq m-1$ we have

$$y_j \quad = \quad n_{j+1} \ + \ \frac{1}{y_{j+1}} \ .$$

Notice that at the final step, where $x_{m+1} = 0$, we simply have $y_m = n_{m+1}$.

Proof of the proposition. This turns out to be a fairly direct consequence of the Euclidean long division result for positive integers: If $0 < a \leq b$ where a and b are integers then $b = aq + r$ where q is a positive integer and $0 \leq r < p$. Suppose now that we are given a positive rational number

$$x \;=\; \frac{a}{b} \;<\; 1$$

and consider its reciprocal

$$\frac{1}{x} \;=\; \frac{b}{a} \;=\; q \;+\; \frac{r}{a}\;.$$

If $x = x_j$ in one of the sequences described above, then $n_{j+1} = q$ and $x_{j+1} = r/a$. Assume now, as we obviously may, that a and b have no common integral factors other than ± 1, so that a and b are uniquely determined by x. There are now two possibilities; either $r = 0$ in which case $x_{j+1} = 0$, or else $0 < r < a$ in which case $0 < x_{j+1} < 1$ and we may rewrite it in reduced terms as r'/a', where $r'd = r$ and $a'd = a$ for some positive integer d (possibly $d = 1$). In this second case the numerator of the least terms representation of the positive rational number x_{j+1} is strictly less than the numerator of x_j. Therefore, if $x_{j+1}, \cdots, x_{j+p}$ are definable with each one positive then the sequence of reduced terms numerators $a = u_j, u_{j+1}, \cdots u_{j+p}$ must be strictly decreasing, and this means that $p < a$.

In particular, if we start out with $x_0 = a/b$, then it follows that x_j must be zero for some $j \leq a$ and the recursive process must terminate.∎

Finding continued fraction expressions. This is extremely routine and best illustrated with a couple of examples. We shall use $x_0 = k/5$ for $k = 2, 3, 4$ (the continued fraction expansion for $1/n$ is always just $1/n$).

If $x_0 \;=\; \frac{2}{5}$, then $y_0 = \frac{5}{2} = 2 + \frac{1}{2}$, so

$$\frac{2}{5} \;=\; \frac{1}{2 + \frac{1}{2}}\;.$$

If $x_0 \;=\; \frac{3}{5}$, then $y_0 = \frac{5}{3} = 1 + \frac{2}{3}$, so that $x_1 = \frac{2}{3}$ and $y_1 = \frac{3}{2} = 1 + \frac{1}{2}$. Therefore

$$\frac{3}{5} \;=\; \frac{1}{1 + \frac{2}{3}} \;=\; \frac{1}{1 + \frac{1}{\frac{3}{2}}} \;=\;$$

$$\frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}\;.$$

Finally, if $x_0 \;=\; \frac{4}{5}$, then $y_0 = \frac{5}{4} = 1 + \frac{1}{4}$, so

$$\frac{4}{5} \;=\; \frac{1}{1 + \frac{1}{4}}\;.$$

Clearly we can reverse this process. For example, suppose that we want to fine the rational number x_0 for which the continued fraction expression is given by $n_1 = 1, n_2 = 2, n_3 = 3$. To find this number, we note that $3 = n_3 = y_2$, so that $x_2 = \frac{1}{3}$, and hence $y_1 = n_2 + x_2 = 2 + \frac{1}{3} = \frac{7}{3}$, so that $x_1 = \frac{3}{7}$, and similarly $y_0 = n_1 + x_1 = 1 + \frac{3}{7} = \frac{10}{7}$, so that finally $x_0 = \frac{7}{10}$.

We can do this algorithmically as follows: Suppose that $x_k = a_k/b_k$. Then we have

$$x_{k-1} \;=\; \frac{1}{n_k + \frac{a_k}{b_k}} \;=\; \frac{b_k}{n_k b_k + a_k}$$

so we have the reverse recursive formulas $a_{k-1} = b_k$ and $b_{k-1} = n_k b_k + a_k$. The reverse recursive process begins with $x_{m-1} = 1/n_m$.