4.D. Conic sections and their coordinate equations

It is not immediately obvious that the various descriptions of conics are equivalent, and it is not necessarily easy to find a reference indicating why the different descriptions yield the same objects. The purpose of this document is to show that the classical Greek view of conic sections — as intersections of a cone with a plane — is related to the currently standard view of conics as curves defined by quadratic equations in the coordinates.

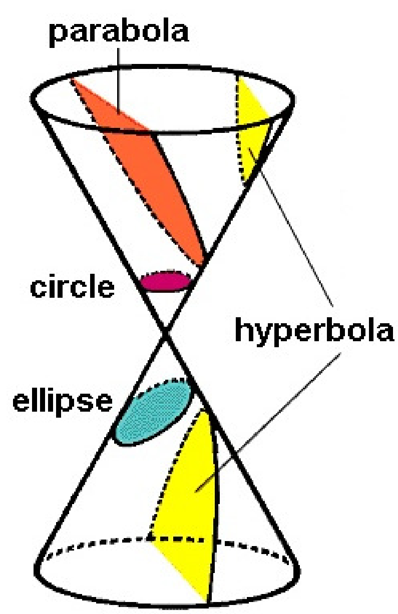
Some of the material below is taken from the following two sources:

http://www.chemistrydaily.com/chemistry/Conic_section

https://en.wikipedia.org/wiki/Conic_section

(Apparently the first link is no longer available.)

We begin with the figure below to illustrate how the different conic sections are formed by intersecting a cone with a plane.



(***Source:*** https://i.pinimg.com/736x/c3/fe/f2/c3fef2c537d36a6bd4ee835d7e684abc.jpg)

Derivations of the conic section equations

Take a cone whose axis is the $z-$ axis and whose vertex is the origin. The equation for the cone is

$$x^2 + y^2 - a^2 z^2 = 0 \qquad (1)$$

where

and θ is the angle which the generators of the cone make with respect to the axis. Notice that this cone is actually a pair of cones, with one cone standing upside down on the vertex of the other cone — or, as mathematicians say, it consists of two "nappes" (pronounced NAPS).

Now take a plane with a slope running along the x direction but which is level along the $y-$ direction. Its equation is

$$z = mx + b \qquad (2)$$

where

$$m = \tan\phi > 0$$

and ϕ is the angle of the plane with respect to the $xy-$ plane.

We are interested in finding the intersection of the cone and the plane, which means that equations **(1)** and **(2)** should be combined. Both equations can be solved for z and we can then equate the two values of z. Solving equation **(1)** for z yields

$$z = \sqrt{\frac{x^2+y^2}{a^2}}$$

and therefore

$$\sqrt{\frac{x^2+y^2}{a^2}} = mx + b.$$

Square both sides and expand the squared binomial on the right side:

$$\frac{x^2+y^2}{a^2} = m^2x^2 + 2mbx + b^2.$$

Grouping by variables yields

$$x^2\left(\frac{1}{a^2} - m^2\right) + \frac{y^2}{a^2} - 2mbx - b^2 = 0. \qquad (3)$$

Note that this is the equation of the projection of the conic section on the $xy-$ plane, hence contracted in the x direction compared with the shape of the conic section itself.

Equation of the parabola

The parabola is obtained when the slope of the plane is equal to the slope of the generators of the cone. When these two slopes are equal, then the angles θ and ϕ become complementary. This implies that

$$\tan\theta = \cot\phi$$

and therefore

$$m = \frac{1}{a} \qquad (4)$$

Substituting equation **(4)** into equation **(3)** makes the first term in equation **(3)** vanish, and the remaining equation is

$$\frac{y^2}{a^2} - \frac{2}{a}bx - b^2 = 0.$$

Multiply both sides by a^2**:**

$$y^2 - 2abx - a^2b^2 = 0$$

Now solve for x**:**

$$x = \frac{1}{2ab}y^2 - \frac{ab}{2}. \qquad (5)$$

Equation **(5)** describes a parabola whose axis is parallel to the $x-$ axis**.** Other versions of equation **(5)** can be obtained by rotating the plane around the $z-$ axis**.**

Equation of the ellipse

An ellipse arises when the sum of the angles θ and ϕ is less than a right angle**:**

$$\theta + \phi < \frac{\pi}{2} \qquad \text{(ellipse)}$$

This means that the tangent of the sum of these two angles is positive**:**

$$\tan(\theta + \phi) > 0.$$

Using the trigonometric identity

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

we see that the condition on the two angles may be written as follows**:**

$$\tan(\theta + \phi) = \frac{m + a}{1 - ma} > 0 \qquad (6)$$

But $m + a$ is positive**,** since the summands are given to be positive**,** so inequality **(6)** is positive if the denominator is also positive**:**

$$1 - ma > 0. \qquad (7)$$

From inequality **(7)** we can deduce

$$ma < 1, \quad m^2a^2 < 1, \quad 1 - m^2a^2 > 0,$$

$$\frac{1}{m^2a^2} > 1, \quad \frac{1}{m^2a^2} - 1 > 0, \quad \frac{1}{a^2} - m^2 > 0 \qquad \text{(ellipse)}.$$

Let us start out again from equation **(3):**

$$x^2\left(\frac{1}{a^2} - m^2\right) + \frac{y^2}{a^2} - 2mbx - b^2 = 0, \qquad (3)$$

This time the coefficient of the x^2 − term does not vanish but is instead positive. Solving for y we obtain the following:

$$y = a\sqrt{b^2 + 2mbx - x^2\left(\frac{1}{a^2} - m^2\right)} \qquad (8)$$

This would clearly describe an ellipse were it not for the second term under the radical, the $2mbx$. It would be the equation of a circle which has been stretched proportionally along the directions of the x − axis and the y − axis. Equation **(8)** is an ellipse but it is not obvious, so it will be rearranged further until this is obvious. Complete the square under the radical, so that the equation transforms into

$$y = a\sqrt{b^2 - \left[x\sqrt{\frac{1}{a^2} - m^2} - \frac{b}{\sqrt{\frac{1}{a^2m^2} - 1}}\right]^2 + \left(\frac{b^2}{\frac{1}{a^2m^2} - 1}\right)}.$$

Now group together the b^2 terms:

$$y = a\sqrt{b^2\left(1 + \frac{1}{\frac{1}{a^2m^2} - 1}\right) - \left[x\sqrt{\frac{1}{a^2} - m^2} - \frac{b}{\sqrt{\frac{1}{a^2m^2} - 1}}\right]^2}.$$

Next, divide by a then square both sides:

$$\frac{y^2}{a^2} + \left(x\sqrt{\frac{1}{a^2} - m^2} - \frac{b}{\sqrt{\frac{1}{a^2m^2} - 1}}\right)^2 = b^2\left(1 + \frac{1}{\frac{1}{a^2m^2} - 1}\right)$$

The x − term in the preceding expression has a mildly complicated coefficient, and it will be useful to pull it out by factoring it out of the second term, which is a square:

$$\frac{y^2}{a^2} + \left(\frac{1}{a^2} - m^2\right)\left(x - \frac{b}{\sqrt{\left(\frac{1}{a^2m^2} - 1\right)\left(\frac{1}{a^2} - m^2\right)}}\right)^2 = b^2\left(1 + \frac{1}{\frac{1}{a^2m^2} - 1}\right)$$

Further rearrangement of constants finally leads to

$$\frac{y^2}{1 - a^2m^2} + \left(x - \frac{mb}{\frac{1}{a^2} - m^2}\right)^2 = \frac{a^2b^2}{(1 - a^2m^2)^2}.$$

The coefficient of the y − term is positive (for an ellipse). Renaming of coefficients and constants leads to

$$\frac{y^2}{A} + (x - C)^2 = R^2 \qquad (9)$$

which is clearly the equation of an ellipse. That is, equation **(9)** describes a circle of radius R and center $(C, 0)$ which is then stretched vertically by a factor of sqrt (A). The second term on the left side (the x − term) has no coefficient but is a square, so that it must be positive. The radius is a product of squares, so it must also be positive. The first term on the left side (the y − term) has a coefficient which is positive, and hence the equation describes an ellipse.

Equation of the hyperbola

The hyperbola arises when the angles θ and ϕ add up to an obtuse angle, which is greater than a right angle. The tangent of an obtuse angle is negative. All the inequalities which were valid for the ellipse become reversed. Therefore we have

$$1 - a^2 m^2 < 0 \qquad \text{(hyperbola)}.$$

Otherwise the equation for the hyperbola is the same as equation **(9)** for the ellipse, except that the coefficient A of the $y-$ term is negative.