Problems from Burton, p. 231 (7th Ed)

7. Follow the hint. Find x & $x+1$ s.t. the square of either plus the sum of both is a square.

Since $x^2 + (x+1) + x = (x+1)^2$,

one condition is satisfied.

But we also want

$$(x+1)^2 + (x+1) + x = (x+a)^2$$

some a, so that

$$4x + 2 = 2ax + a^2.$$

[note the LHS is $x^2 + 4x + 2$]

Hence $x(4-2a) = a^2 - 2$ or

$x = \dfrac{a^2-2}{4-2a}$. This will be positive if

$2 < a^2 < 4$ and hence we can solve it if we take a s.t. $\sqrt{2} < a < 2$. It is also positive if $a < -\sqrt{2}$, and the answer on page 768 of Burton uses $a = -2$, getting $x = \frac{1}{4}$ and $(x+1) = \frac{5}{4}$.

9. Let x, $4x+4$, 1 be given, so that

$$x^2 + (4x+4)\cdot 1 = (x+2)^2$$
$$1^2 + x(4x+4) = (2x+1)^2.$$

We want to find an x so that

$$(4x+4)^2 + x\cdot 1 = 16x^2 + 33x + 16$$

is a perfect square, say $(4x+a)^2 = 16x^2 + 8ax + a^2$. Equating both expressions yields

$$33x + 16 = 8ax + a^2, \text{ so that}$$

$$(33-8a)x = a^2 - 16 \quad \text{or} \quad x = \frac{a^2-16}{33-8a}.$$

As before, one can solve this if $a < -4$ and get a positive value of x. The book takes $a = -5$ and gets the corresponding values

$$x = \frac{9}{73} \qquad 4x+4 = \frac{36 + 4\cdot 73}{73} = \frac{328}{73}$$
$$1 = 1$$

Notice that if we multiply these numbers by a constant K, they have the property in the problem. In particular, this works for $9, 73, 328$.

11. By the hint in Burton, if $a = 8x$ and $b = x^2 - 1$, then $ab + a = (2x)^3$. We want to find x so that $ab + b = 8x(x^2-1) + (x^2-1)$ $= 8x^3 + x^2 - 8x - 1$ is a perfect cube, say $(2x-c)^3 = 8x^3 - 12cx^2 + 6c^2x - c^3$.

Equating the expressions, we obtain

$$8x^3 + x^2 - 8x - 1 = 8x^3 - 12cx^2 + 6c^2x - c^3$$

and if we subtract the right hand side from both sides, we get

$$(12c+1)x^2 - (8+6c^2)x + (c^3-1) = 0.$$

A typical trick that Diophantus uses at this point is to choose c so that the constant term vanishes. This means $c = 1$.

With this choice we have

$$13x^2 - 14x = 0, \text{ so that}$$

$$x = \frac{14}{13}.$$

Substituting this into the formulas for a and b, we find that

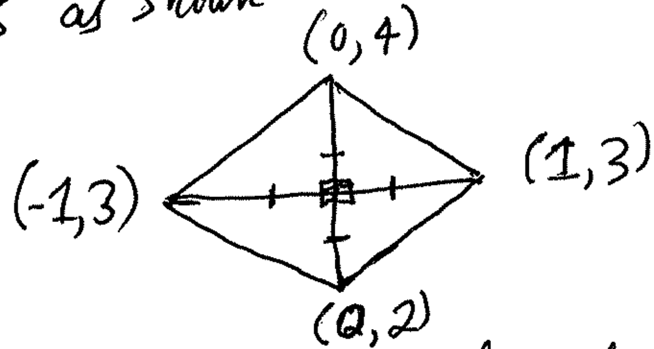
$$a = 8x = \frac{8 \cdot 14}{13} = \frac{112}{13},$$

$$b = x^2 - 1 = \frac{14^2 - 13^2}{13^2} = \frac{27}{169}.$$

Note: One can get a negative rational solution by setting $12c+1=0$, solving for x (the equation becomes linear), and then substituting this value into the expressions for a and b.

An application of Pappus' Centroid Thm:

Let A be the region in the coordinate plane bounded by the parallelogram with vertices as shown:



What is the volume of the solid of revolution formed by rotating A about the x-axis?

<u>SOLUTION</u>: The area is 4 times the area of ~~each~~ one triangle in the picture, so it is 2. The centroid is $(0,3)$. Therefore the volume equals

$$2\pi \cdot \underset{\substack{\uparrow \\ \text{centroid} \\ \text{to} \\ \text{x-axis}}}{3} \cdot \underset{\substack{\uparrow \\ \text{area}(A)}}{2} = 12\pi.$$

(No need to evaluate any definite integrals!)