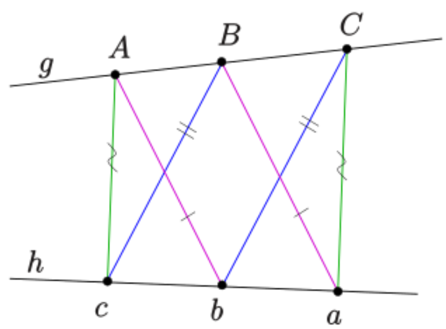
5.G. THE PAPPUS HEXAGON THEOREM

This result of Pappus anticipated several developments in geometry beginning in the 15th century. There are several versions of this result; we shall begin with a special case called the ***affine form***. We are including it here to illustrate one way in which Pappus' original results went beyond classical Euclidean geometry.

Pappus' Hexagon Theorem, affine form. *Suppose that* **{A, B, C}** *and* **{a, b, c}** *are two triples of collinear points in the plane, and assume that the two lines are distinct. If the lines* **Ab** *and* **aB** *are parallel and the lines* **Bc** *and* **bC** *are parallel, then the lines* **Ac** *and* **aC** *are also parallel.*



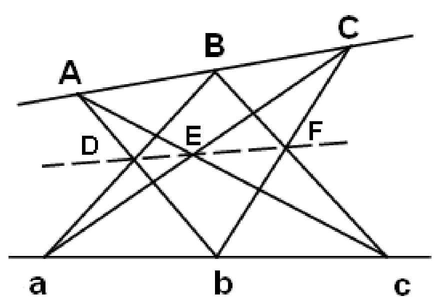
(*Source:* https://en.wikipedia.org/wiki/Pappus%27s_hexagon_theorem)

Neither the hypothesis nor the conclusion of this result involves measurement concepts such as distance (or length of segments) and angle measurement, but in order to prove it one needs some relatively weak algebraic input related to measurements. One proof of the theorem as stated above is given in the previously cited ***Wikipedia*** article.

There are also versions of the Pappus Hexagon Theorem if either **Ab** is not parallel to **aB** or **Bb** is not parallel to **bC** (or both). If neither either **Ab** not parallel to **aB** nor **Bb** is parallel to **bC** then the result takes the following form:

Pappus' Hexagon Theorem, second form. *Suppose that* **{A, B, C}** *and* **{a, b, c}** *are two triples of collinear points in the plane, and assume that the two lines are distinct. If the lines* **Ab** *and* **aB** *meet at a point* **F** *and the lines* **Bc** *and* **bC** *meet at a point* **D***, then* **either** *the lines* **Ac** *and* **aC** *meet at a point* **E** *such that* **D, E, F** *are collinear* **or else** *the three lines* **Ac***,* **aC** *and* **DF** *are all parallel to each other.*

Here is a drawing for the case in which there are no parallel pairs:



Finally, here is a version of the result if one of the line pairs **{Ab, aB}** and **{Bc, bC}** is a parallel pair and the other is not.

<u>Pappus' Hexagon Theorem, third form.</u> *Suppose that* **{A, B, C}** *and* **{a, b, c}** *are two triples of collinear points in the plane, and assume that the two lines are distinct. If the lines* **Ab** *and* **aB** *meet at a point* **F** *and the lines* **Bc** *and* **bC** *are parallel, then the lines* **Ac** *and* **aC** *meet at a point* **E** *such that the three lines* **Bc**, **bC** *and* **DE** *are all parallel to each other.*

There is a unified setting in which one can prove all three forms of the Pappus Hexagon Theorem simultaneously. Here are three references:

http://math.ucr.edu/~res/math133-2018/geometrynotes04a.f13.pdf

http://math.ucr.edu/~res/math133-2018/geometrynotes04b.f13.pdf

http://math.ucr.edu/~res/progeom/pg-all.pdf

More precisely, this can be done by combining Sections **IV.1 – IV.3** of the first document with Section **IV.5** of the second, or by combining Chapter **III** with Section **V.3** in the third.