Magic squares uniqueness proof

The goal is to show that there is only one **3** by **3** magic square with entries **1** through **9**, up to rotation and reflection**.**

By definition a **3** by **3** matrix is a magic square if the rows**,** columns and diagonals all add up to the same number**.** The following general considerations are taken from the website

http://home.earthlink.net/~morgenstern/magic/sq3.htm .

Three-Term Formulation

Suppose we have a 3x3 magic square.

$$\begin{matrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{matrix}$$

Any 3x3 magic square can be expressed using the three terms,

$a = (E_{23}-E_{31})$,
$b = (E_{21}-E_{33})$, and
$c = E_{22}$.

Proof

$E_{21}+E_{22}+E_{23} = E_{11}+E_{21}+E_{31} \implies E_{11} = E_{22}+(E_{23}-E_{31}) = c+a$.
$E_{13}+E_{23}+E_{33} = E_{13}+E_{22}+E_{31} \implies E_{33} = E_{22}-(E_{23}-E_{31}) = c-a$.
$E_{21}+E_{22}+E_{23} = E_{13}+E_{23}+E_{33} \implies E_{13} = E_{22}+(E_{21}-E_{33}) = c+b$.
$E_{11}+E_{21}+E_{31} = E_{11}+E_{22}+E_{33} \implies E_{31} = E_{22}-(E_{21}-E_{33}) = c-b$.
$E_{11}+E_{12}+E_{13} = 3E_{22} \implies (E_{22}+a)+E_{12}+(E_{22}+b) = 3E_{22} \implies E_{12} = c-(a+b)$.
$E_{31}+E_{32}+E_{33} = 3E_{22} \implies (E_{22}-b)+E_{32}+(E_{22}-a) = 3E_{22} \implies E_{32} = c+(a+b)$.
$E_{11}+E_{21}+E_{31} = 3E_{22} \implies (E_{22}+a)+E_{21}+(E_{22}-b) = 3E_{22} \implies E_{21} = c-(a-b)$.
$E_{13}+E_{23}+E_{33} = 3E_{22} \implies (E_{22}+b)+E_{23}+(E_{22}-a) = 3E_{22} \implies E_{23} = c+(a-b)$.

Substituting a,b,c for the E's:

c+a	c-(a+b)	c+b
c-(a-b)	c	c+(a-b)
c-b	c+(a+b)	c-a

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Any values for a,b,c make a 3x3 magic square.
The sum of each row, column, and diagonal is 3c.
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Suppose now that we want the entries of the magic square to be **1, 2, 3, 4, 5, 6, 7, 8, 9.** Then the sum of all the entries is $\mathbf{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45.}$ On the other hand, by the preceding paragraph the sum of the entries in each row is **3c,** so the sum of the entries in all the rows is **9c,** which means that $\mathbf{c = 5.}$

The next question is where to put **1.** By the symmetry properties of the square there are essentially two possibilities: ***Either*** there is a **1** in one of the corner positions ***or else*** there is a **1** in one of the **(1, 2), (2, 1), (2, 3)** or **(3, 2)** entries. Suppose first that the **1** is in a corner position; then by symmetry we can put **1** in the lower right position. Since the **NW** to **SE** diagonal must add up to **15,** it follows that there must be a **9** in the upper left position. In the general setting, we then have $\mathbf{c = 5}$ and $\mathbf{a = 4,}$ so the remaining entries are given as follows:

9	4-b	5+b
4+b	5	9-b
5-b	6+b	1

Now the first row is supposed to add up to **15,** but clearly it adds up to **18.** This implies that we cannot have a magic square in which **9** and **1** are in opposite corners, and in particular **1** must appear in of the **(1, 2), (2, 1), (2, 3)** or **(3, 2)** entries. By symmetry we might as well assume that **1** appears in the **(1, 2)** entry, and as before it follows that **3** must appear in the **(1, 2)** entry. In the notation at the top of the page, this means that $\mathbf{a + b = 4,}$ which in turn implies that $\mathbf{a = 4 - b}$ and $\mathbf{a - b = 4 - 2b.}$ If we substitute these conclusions into the matrix at the top of the previous page, we see that the magic square must have the following form:

9-b	1	5+b
1+2b	5	9-2b
5-b	9	1+b

Once again we know that the sum of the entries in the bottom row equals **15,** and the only way this can happen if the entries are **1 – 9** is if one entry in the top row equals **2** and the other equals **4** (the sum of the entries in the third row is **15,** so the sum of the left and right entries in the third row is **6;** these entries are unequal**,** and neither can be equal to **1** because we know that **1** appears elsewhere in the matrix)**.** Once again by symmetry we may as well assume that lower left entry is **4** and the other is **2.** But these conditions imply that **b** = **1,** and if we substitute this value into the remaining terms of the magic square we obtain

8	1	6
3	5	7
4	9	2

which is exactly the magic square in the previous document**.** Thus we have shown that every **3** by **3** magic square whose entries are **1 – 9** can be transformed to the original example by a (possibly empty) sequence of rotations and reflections**.**

Additional references

There is an extensive literature dealing with **3** by **3** and larger magic squares like the following example**,** which appears in Albrecht Dürer's engraving ***Melencolia I*:**

16	3	2	13
5	10	11	8
9	6	20	3
4	10	14	22

(Source: http://stickypix.net/up/files/21477_f4psw/Magic%20Squares.jpg)

A more detailed discussion of magic squares is beyond the scope of this course**,** but here are two online references**:**

http://en.wikipedia.org/wiki/Magic_square

http://mathworld.wolfram.com/MagicSquare.html