Pythagorean triples and squares in arithmetic progressions

These are two topics that Fibonacci studied, but he was (apparently) not aware of their close relationship.

Pythagorean Triple	Squares in Progression	Relationship
		$b = z$
$x^2 + y^2 = z^2$	$b^2 - a^2 = c^2 - b^2$	$c = x + y$
$x < y$	common difference $= d$	$a = y - x$

check these are equivalent to the formulas on page 5 of history07b.pdf (course directory).

<u>Example</u> $(x, y, z) = (3, 4, 5)$. Then $(a, b, c) = (1, 5, 7)$ and $5^2 - 1^2 = 24 = 7^2 - 5^2$.

Note that $d = 2xy = 24$.

More examples

Pythagorean triples

Start with $(k+1)^2 = k^2 + 2k + 1$.

If $2k+1$ is a perfect square, say $(2m+1)^2$, then $(2m+1, k, k+1)$ is a Pythagorean triple; in fact we can write $k = 2(m^2+m)$.

Examples

$x = \sqrt{2k+1}$	$y = k$	$z = k+1$
3	4	5
5	12	13
7	24	25
9	40	41
11	60	61

etc.

Let's use these to find triples of squares in an arithmetic progression.

m	x	y	z	a	b	c	d
1	3	4	5	1	5	7	24
2	5	12	13	7	13	17	120
3	7	24	25	17	25	31	336
4	9	40	41	31	41	49	720
5	11	60	61	49	61	71	1320

<u>Notice the pattern</u>

By construction $x_m = 2m+1$, $y_m = 2m(m+1)$, $z_m = y_m + 1$, $b_m = z_m$, $a_m = y_m - x_m$, $c_m = y_m + x_m$. The table suggests $c_m = a_{m+1}$. <u>Prove it!</u>

<u>Here is the proof:</u>

$$a_m = \underset{y_m}{(2m^2+2m)} - \underset{x_m}{(2m+1)} = 2m^2-1$$

$$C_m = (2m^2+2m) + (2m+1) =$$

$$2m^2+4m+1 = 2(m+1)^2-1 =$$

$$a_{m+1} .$$