8. Mathematics in the late Middle Ages

(Burton, **7.1, 7.2;** some material not in Burton)

The transition from ancient to modern mathematics period began with the breakthroughs of Indian mathematicians and continued with the work of Arabic mathematics. Both of the latter remained productive through much of the second half of this transitional period, which roughly covers the time from 1200 to 1600. In the preceding unit we discussed the beginning of the second half of the transitional period, during which there was a revival of activity in Christian Western Europe, and we shall continue the narrative here.

General remarks about late medieval and Renaissance mathematics

Many histories of mathematics view the time after Fibonacci until the early sixteenth century as a period of decline and inactivity. While the work during that period did not contain any advances at the level of Fibonacci's work, there were some modest but important developments involving mathematics that took place during that time. Many can be placed into the following two categories:

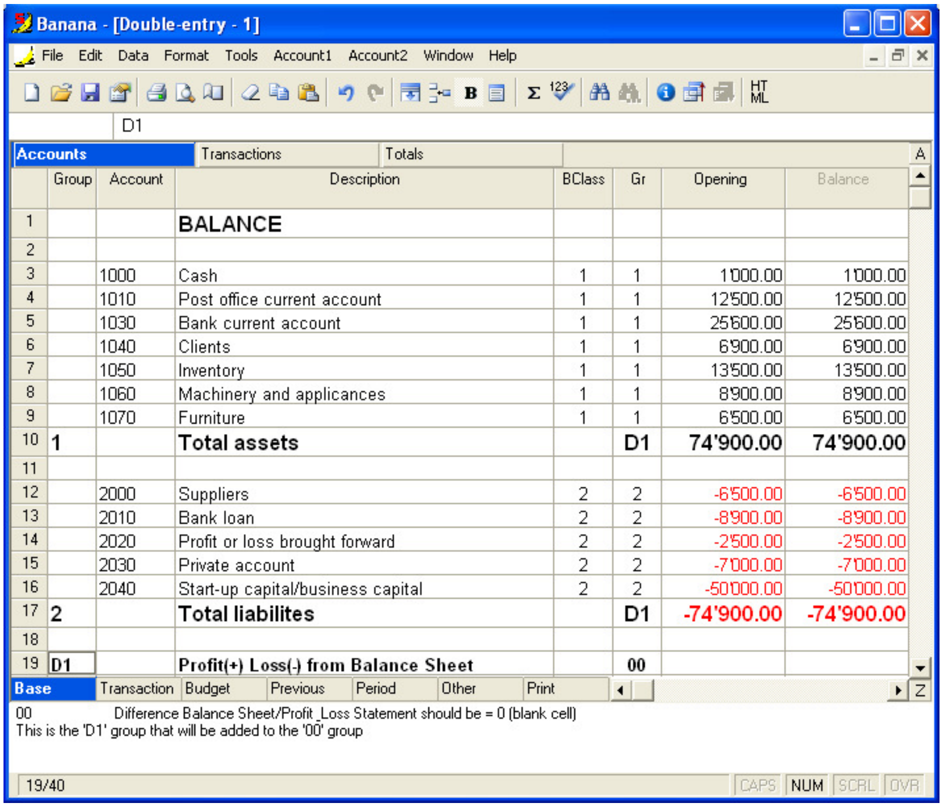
1. Improvements in mathematical notation.
2. Stronger ties to science, the arts and commerce.

Popularization of base 10 *arithmetic and other notational improvements*

Increased trade in the Mediterranean area was an early sign of recovery from the Dark Ages, and Italy was a center of this trade. During the 11th and 12th century, several developments contributed substantially to increased trading activity; these include the Christian conquests discussed in the previous unit and the Crusades, which began at the end of the 11th century. Fibonacci came from a merchant family, and commercial ventures brought him into contact with the mathematical activity in Islamic lands. One clear reason for his interest in mathematics was its potential usefulness in handling the steadily more complicated counting and accounting problems generated by the increasingly active commercial trade of the time. Business was moving from being a sequence of separate, relatively self – contained transactions to overlapping inward and outward flows of merchandise and money. Well before the beginning of the 14th century, it became necessary to manage items like financial credit, debts, interests, and transaction records. In particular, these needs led to the gradual adoption of **double entry bookkeeping**, which kept records of where money was coming from and where it was going; fragmentary records indicate that such systems were already in use early in the 13th century, and by the middle of the 14th century they were very widely used. The underlying logic of double entry bookkeeping is summarized in a simple equation:

Assets = Liabilities + Owner's Equity

In other words, what a business owns must always equal **(=)** what it owes to its creditors plus **(+)** what it owes to the owner or owners. An example is given on the next page.



	Group	Account	Description	BClass	Gr	Opening	Balance
1			BALANCE				
2							
3		1000	Cash	1	1	1'000.00	1'000.00
4		1010	Post office current account	1	1	12'500.00	12'500.00
5		1030	Bank current account	1	1	25'600.00	25'600.00
6		1040	Clients	1	1	6'900.00	6'900.00
7		1050	Inventory	1	1	13'500.00	13'500.00
8		1060	Machinery and applicances	1	1	8'900.00	8'900.00
9		1070	Furniture	1	1	6'500.00	6'500.00
10	1		**Total assets**		D1	**74'900.00**	**74'900.00**
11							
12		2000	Suppliers	2	2	-6'500.00	-6'500.00
13		2010	Bank loan	2	2	-8'900.00	-8'900.00
14		2020	Profit or loss brought forward	2	2	-2'500.00	-2'500.00
15		2030	Private account	2	2	-7'000.00	-7'000.00
16		2040	Start-up capital/business capital	2	2	-50'000.00	-50'000.00
17	2		**Total liabilites**		D1	**-74'900.00**	**-74'900.00**
18							
19	D1		**Profit(+) Loss(-) from Balance Sheet**		00		

(Source: http://www.banana.ch/accounting/eng/images/double_entry_01.png)

The knowledge that Italian merchants required was not readily available from courses of study from the church or from the universities of that time. This need for instruction led to the emergence of a new class of mathematicians, who provided instruction to the merchants and wrote texts from which they taught the requisite material. There schools were called Abacus Schools, and the instructors were known as ***Maestri d'abbaco*** (***Note:*** Despite this name, the schools taught pen – and – paper computations using Hindu – Arabic numerals ***without*** the use of a calculating device such as an abacus; some authors call these instructors ***abacists***, but usually the latter term denotes those who favored computations with devices like an abacus and strongly opposed the methods taught in the abacus schools — in fact, the instructors at the Abacus Schools were from a rival group called ***algorists***.). As trade and commerce grew during the 14th through 16th century, similar classes of masters of mathematics came into being, and similar schools appeared in other European countries. For example, this took place in Germany on a fairly extensive scale during the middle to late 15th century.

Large numbers of such texts have been preserved, and by the early 14th century some of them (for example, the 1307 work, *Tractatus algorismi*, by Jacopo of Florence) had progressed significantly beyond Fibonacci in some respects. Several of these Italian mathematicians of the 14th century were instrumental in teaching merchants the "new" Hindu – Arabic decimal place – value system and the algorithms for using it. There was formidable opposition to the new numbering system and computational techniques, both in Italy and the rest of Europe; one substantive reason opposition was that the ten symbols for Hindu – Arabic numbers did not become standardized for some time, and this led to obvious problems with (mis)understanding the numbers they represented. However, the new methods, which were of course more efficient and convenient to use once they were mastered, eventually became the accepted standard, first in Italy and later throughout the rest of Europe.

The Italians were thoroughly familiar with Arabic mathematics and its emphasis on algebraic methods. Although their teaching focused on practical business problems they also studied various recreational problems, including examples in geometry, elementary number theory, the calendar, and astrology. In connection with their instructional and recreational mathematics, in some cases they extended the Arabic methods by introducing *abbreviations and symbolisms*, developing new methods for dealing with complex algebraic problems and allowing the use of symbols for unknowns. Thus, unlike Arabic algebra, which was entirely rhetorical, the algebra of the Italians frequently used syncopated notation to varying degrees, and this became more widespread as the 14th century progressed.

Another innovation was the extension of the Arabic techniques for solving quadratic equations to higher degree polynomials. For example, a 1344 book by Maestro Dardi of Pisa extended al-Khwarizmi's standard list of **6** types of quadratic equations to a list of **198** equations of degree less than or equal to **4,** and he gave a method for solving one type of cubic equation. Unfortunately, as with several other works of this era, his *Aliabraa – Argibra* contains incorrect formulas along with its noteworthy advances; one particularly comprehensive and reliable work summary from the period was the *Trattato di praticha d'arismetrica*, by Maestro Benedetto of Florence (1429 – 1479).

Here is an example of one problem from this era, which was formulated by Antonio de' Mazzinghi (1353 – 1383), one of the most highly regarded mathematicians from the period: *Find two numbers such that multiplying one by the other yields* **8** *and the sum of their squares is* **27.** — The solution begins by supposing that the first number is one number minus the root of some other number, while the second number equals one number minus the root of some other number. The problem leads to the equations

$$(x-\sqrt{y})(x+\sqrt{y}) = 8$$
$$(x-\sqrt{y})^2+(x+\sqrt{y})^2 = 27$$

for which the solution is given by

$$x = \frac{\sqrt{43}}{2}, \quad y = \frac{11}{4}.$$

Most of the standard textbook discussions of the Italian masters of mathematics and their work are extremely brief. There are more extensive surveys of this period in Section 3 of the online article and pages 42 – 52 of the book cited below:

http://www.melaracconto.org/algebra/algebra/storia/siti/The%20Art%20of%20Algebra%20by%20Karen%20H_%20Parshall.htm

van der Waerden, B. L. A history of algebra. From al-Khwārizmī to Emmy Noether. *Springer-Verlag, Berlin,* 1985.

Here are online references for even more detailed accounts of the era (unfortunately, not in English but adequately translatable using standard online software):

http://akira.ruc.dk/~jensh/Publications/2008_Ueber%20den%20italienischen%20Hintergrund.pdf

http://php.math.unifi.it/archimede/archimede/fibonacci/catalogo/ulivi.php

The mathematical theory of perspective drawing

[*Note*: This material is not covered in Burton. **]**

As the cultural and commercial center of the late Middle Ages and early Renaissance, Italy was the source of many important new developments during that period. In particular, an important change took place in painting around the year 1300. Prior to that time the central objects of paintings were generally flat and more symbolic than real in appearance; emphasis was on depicting religious or spiritual truths rather than the real, physical world. As society in Italy became more sophisticated, there was an increased interest in using art to depict a wider range of themes and to so in a manner that more accurately captured the image that the human eye actually sees, and recent (at the time) translations of Alhazen's work on optics yielded scientific and mathematical principles for creating such paintings.

The earlier concepts of art are clearly represented in a segment from the famous Bayeux Tapestry which is a graphic account of the Norman conquest of England in 1066. In the segment depicted at the online site as **http://hastings1066.com/bayeux23.shtml** there are men eating at a table, and it looks as if the objects on the table are directly facing the viewer and ready to fall off the table's surface. In this and other segments of the tapestry one can also notice the flat appearance of nearly all objects. None of this detracts from the artistic value of the tapestry and some of the distortion can be explained because this is a tapestry rather than a painting, but medieval paintings also have many of the same traits. The following link contains a painting with a similar example involving tables whose tops appear as if they might be vertical.

http://www.mcm.edu/academic/galileo/ars/arshtml/renart1.html

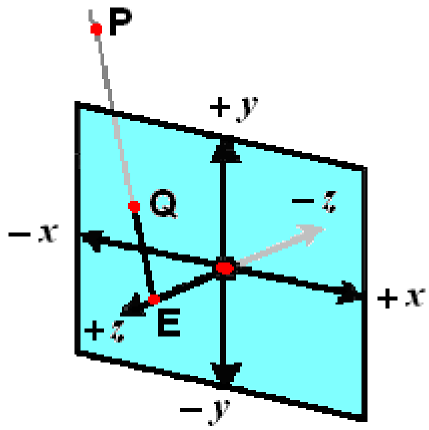
And here is one more example titled *Woman teaching geometry*, which appears at the beginning of a medieval translation of Euclid's ***Elements*** from around 1310.

(***Source:*** **http://en.wikipedia.org/wiki/Euclid's_Elements**)

In this picture, one can also notice the flat appearance of nearly all the faces.

The paintings of Giotto (Ambrogio Bondone, 1267 – 1337), especially when compared to those of his predecessor Giovanni Cimabue (originally Cenni di Pepo, 1240 – 1302), show the growing interest in visual accuracy quite convincingly, and other painters from the 14th and early 15th century followed this trend. It is interesting to look at these paintings and see how the artists succeeded in showing things accurately much of the time but were far from perfect. Eventually artists with particularly good backgrounds in Euclidean geometry began a systematic study of the whole subject and developed a mathematically precise theory of perspective drawing.

The basic idea behind the theory of perspective is illustrated below. Assume that the eye **E** is at some point on the positive $z-$axis, the canvas is the $xy-$plane, and the object **P** to be included in the painting is at the point **P** which is on the opposite side of the $xy-$plane as the eye **E.** Then the image point **Q** on the canvas will be the point where the line **EP** meets the $xy-$plane.



A detailed analysis of this geometrically defined mapping yields numerous facts that are logical consequences of the construction and Euclidean geometry. For example, one immediately has the following conclusion.

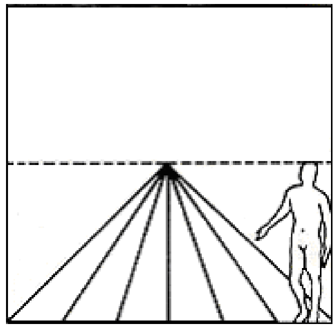
<u>**PROPOSITION.**</u> *If* **P**, **P′** *and* **P″** *are collinear points on the opposite side of* **E** *and* **Q**, **Q′** *and* **Q″** *are their perspective images, then* **Q**, **Q′** *and* **Q″** *are also collinear.*

<u>***Proof.***</u> Let **L** be the line containing the three points. Then there is a plane **A** containing **L** and the point **E;** the three points **Q**, **Q′** and **Q″** all lie on the intersection of **A** with the $xy-$plane. Since the intersection of two planes is a line it follows that the three points must lie on this line.■

Further analysis yields the following important observation; a proof of this statement is described in **(8.D).**

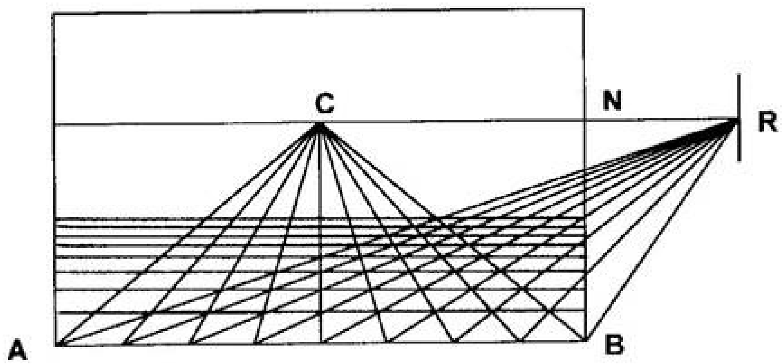
<u>**VANISHING POINT PROPERTY.**</u> *If* **L**, **M** *and* **N** *are mutually parallel lines, then their perspective images pass through a single point on the* $xy-$ *plane. This point is known as the* ***vanishing point.*** *The set of all vamishing points on all lines is the* $x-$ *axis.*

Here is one picture to illustrate the Vanishing Point Property:



(*Source:* http://www.collegeahuntsic.qc.ca/Pagesdept/Hist_geo/Atelier/Parcours/Moderne/perspective.html)

And here is another depicting two different families of mutually parallel lines:



It is enlightening to examine some paintings from the 14th and 15th centuries to see how well they conform to the rules for perspective drawing. In particular, the site

http://www.webexhibits.org/sciartperspective/perspective1.html

analyzes Giotto's painting *Jesus Before the Caïf* (1305) and shows that the rules of perspective are followed very accurately in some parts of the painting but less accurately in others.

The first artist to investigate the geometric theory of perspective systematically was F. Brunelleschi (1377 – 1446), and the first text on the theory was *Della Pittura*, which was written by L. B. Alberti (1404 – 1472). The influence of geometric perspective theory on paintings during the fifteenth century is obvious upon examining works of that period. The most mathematical of all the works on perspective written by the Italian Renaissance artists in the middle of the 15th century was *On perspective for painting (De prospectiva pingendi)* by P. della Francesca (1412 – 1492). Not surprisingly, there were many further books written on the subject at the time, of which we shall only mention the *Treatise on Mensuration with the Compass and Ruler in Lines, Planes, and Whole Bodies*, which was written by Albrecht Dürer (1471 – 1528) in 1525.

Here are additional online references for the theory of perspective with a few examples:

http://mathforum.org/sum95/math_and/perspective/perspect.html

http://www.math.utah.edu/~treiberg/Perspect/Perspect.htm

http://www.dartmouth.edu/~matc/math5.geometry/unit11/unit11.html

Here is a link to a perspective graphic that is animated:

http://gaetan.bugeaud.free.fr/pcent.htm

Of course, one can also use the theory of perspective to determine precisely how much smaller the image of an object becomes as it recedes from the $xy-$ plane, and more generally one can use algebraic and geometric methods to obtain fairly complete quantitative information about the perspective image of an object. Such questions can be answered very systematically and efficiently using computers, and most of the time (if not always) the $3-$ dimensional graphic images on computer screens are essentially determined by applying the rules for perspective drawing explicitly.

The consolidation of trigonometry

Although there is no specific date when the Middle Ages ended and the Renaissance began, the transition is generally marked by three events during the second half of the fifteenth century.

1. The invention of the printing press with movable type in 1452 by Johannes Gutenberg (*c.* 1400 – 1468).
2. The end of the Byzantine Empire with the Ottoman (Turkish) conquest of Constantinople in 1453 by Sultan Mehmet II the Conqueror (1423 – 1481; reigned 1444 – 1446 and 1451 – 1481).
3. The European (re)discovery of America in 1492 by Christopher Columbus (1451 – 1506).

One could also add the end of the conquest of Granada and expulsion of the Moors from Spain in 1492, and for our purposes this is particularly significant because of the Arabic influences on the history of mathematics. The level of mathematical and other scientific activities in such cultures had been declining more or less gradually ever since the 12th century. Although there were still a few noteworthy mathematical contributors during the 15th century, there were virtually none afterwards.

Following the Turkish conquest of Constantinople and other Greek states in the area, many Greek scholars brought manuscripts of ancient Greek writers to Western Europe. These manuscripts led to more accurate and informed translations in many cases. Activity in this direction continued for about a century, culminating in the translations of F. Commandino (1509 – 1575). Although the impact of the movable type printing press for mathematics was not so immediate, it did lead to greatly increased communications among scholars and eventually to wider circulation of new ideas in mathematics; a few early and especially important examples of printed mathematical treatises from the 15th and 16th centuries will be mentioned later in this unit.

New translations played a role in one significant mathematical development during the second half of the 15th century; namely, the emergence of trigonometry as a subject in its own right. Ever since Hellenistic times, trigonometry had been regarded by Greek, Indian and Arabic scientists mainly as a mathematical adjunct to observational astronomy. However, as trigonometry expanded in content and found increasingly many applications to other subjects such as navigation, surveying, and military engineering, it became clear that the subject could no longer be viewed in this fashion. We have already noted that some non – Western mathematicians like Nasireddin and Bhaskara had taken steps towards recognizing that trigonometry was no longer subservient to observational astronomy.

The separation of these subjects was made very explicit in the work of Johann Müller of Königsberg – in – Franconia (1436 – 1476), who is better known as ***Regiomontanus,*** which is a literal Latin translation of Königsberg (***Note*:** Königsberg – in – Franconia is a small town in the northwest corner of the present German state of Bavaria, close to Bamberg on the map **http://www.mapzones.com/citymap/germany/bavaria/bavaria.jpg**, and **NOT** the more famous East Prussian city on the Baltic Sea which is now called Kaliningrad and lies in a small enclave of Russia sandwiched between Poland and Lithuania; see **http://www.inkaliningrad.com/english/wp-content/uploads/2008/03/map-kaliningrad.JPG** for the location of Kaliningrad). Just like many of his contemporaries known as "Renaissance men," he had extremely broad interests and abilities. Regiomontanus made new translations of various classical works, and in his book *De Triangulis (On Triangles)* he organized virtually everything that was known in plane and spherical trigonometry at the time, from the classical Greek and Arabic results to more recent discoveries. In particular, this work systematically develops topics such as the determination of all measurements of a triangle from the usual sorts of partial data (side – angle – side, *etc*.) and states the Law of Sines explicitly. We have already mentioned the 13th century work of Nasireddin in this direction; Regiomontanus' treatment fell short of Nasireddin's in some respects, but it recognized plane trigonometry somewhat more explicitly, and it had a major impact on the development of trigonometry and its ties to astronomy and algebra, partly because it was completed at the right time and in the right place.

In another work, *Tabulæ directionum*, Regiomontanus gives extensive trigonometric tables and introduces the tangent function. To provide an idea of the accuracy of his results, we note that his computations essentially give **57.29796** as the tangent of 89° and the correct value to five decimal places is **57.28896.**

Fifty years ago subjects like solid geometry and spherical trigonometry were standard parts of the high school mathematics curriculum, but since this is no longer the case we shall include some online background references for spherical geometry and basic spherical trigonometry here:

http://www.math.uncc.edu/~droyster/math3181/notes/hyprgeom/node5.html

http://mathworld.wolfram.com/SphericalTrigonometry.html

http://star-www.st-and.ac.uk/~fv/webnotes/chapter2.htm

There is one other contribution by Regiomontanus that we shall mention because it foreshadows some of the major revolutions in scientific thought during the 16th century. We have already noted that there are exactly three ways of decomposing the plane into solid regular polygonal regions; namely, one can do this with regular triangles, squares or hexagons but not with any other sorts of regular polygons. One can pose a similar question for **3** – dimensional space. There is of course the obvious decomposition into cubes, and one can ask if there are any others. The writings of Aristotle (in ***Posterior Analytics***) contain an assertion that this can be done with regular tetrahedra, and various scholars spent a great deal of time and effort in attempts to describe such decompositions. However, it turns out that such decompositions cannot exist, and Regiomontanus is credited with this discovery. Further information and a derivation appear in the supplementary file **(8.B).** Certainly this discovery was not as revolutionary a scientific advance as the later work of N. Copernicus (1473 – 1543) in astronomy, but it is an earlier example of Renaissance scientists' willingness and ability to question established scientific ideas and, in some cases, to correct them (in this connection it is worthwhile to note that Regiomontanus firmly believed in the Ptolemaic view of the universe). This leads directly to the next topic:

New directions in scientific thought

Not surprisingly, the rediscovery of ancient learning during the late Middle Ages and the revival of intellectual activity led to questions about how it should be carried forward. On one side there was interest in using the work of the ancient Greeks to study religious and philosophical issues (this was the theme of **scholasticism**, which dominated medieval thought from the 12th through the 15th centuries). Eventually some of these efforts moved in directions which seem quite bizarre by today's standards (for example, sustained attempts to discover properties of angels using logical deduction; however, despite its frequent repetition ***there is no evidence that scholars actually debated about the number of angels who could dance on the point of a needle***). However, other directions of inquiry were important steps in the evolution of the modern scientific method, building upon the earlier fundamental work of Alhazen. In a closely related vein, there was also interest in putting this knowledge to practical use. Eventually all of these viewpoints found a place in late medieval and Renaissance learning, but the balance was weighted more towards observation and the practical considerations than it had been in Greek culture. One clear manifestation of this in the sciences was the emphasis on systematic experimentation and finding clear, relatively simple explanations for natural phenomena. Mathematical knowledge during the late Middle Ages and Renaissance expanded in response to these increased practical and scientific needs. In a different direction, the writings of Nicholas of Cusa (or Kues, 1401 – 1464) proposed alternatives to Greek natural philosophy and scholasticism which strongly influenced later scientists including Copernicus, Galileo and Kepler. More will be said about his contributions later when we discuss progress which led to the development of calculus.

Note: Nicholas of Cusa's legacy is really extraordinary; although his methods were often unscientific, many of his bold and unconventional conclusions (for example, the earth is not the center of the universe and it is not at rest, while other celestial bodies are not perfectly spherical and do not move in perfectly circular orbits) were shown to be correct one or two centuries later. Furthermore, his willingness to legitimize the concept of infinity went beyond earlier views, and it foreshadowed many future developments.

Advances in mathematical notation

We have already noted the introduction of the Hindu – Arabic numeration system and some progress towards creating more concise ways of putting mathematical material into written form by successors to Fibonacci. Although some abbreviations and symbols had been introduced, only a few abbreviations of Italian words (for example, **cos** for **cosa** = *unknown*) had come anywhere close to being standard notation. However, during the 15th century mathematicians had begun to devise some of the symbols that we use today. The next page has a few examples beyond those already mentioned.

Symbolism	Year	Developer
Fraction bar for numerator over denominator	*c.* 1200	Al-Hassar (12th century) Fibonacci (in ***Liber abaci***)
Juxtaposition for multiplication	15th century 1544	Al-Qalasadi (1412 – 1486) M. Stifel (1487 – 1567)
Superscripts for exponents	1484	N. Chuquet (1445 – 1488)
+ and –	1489	J. Widman (1462 – 1498)
√ (radical sign)	1525	C. Rudolff (1499 – 1545)
=	1557	R. Recorde (1510 – 1558)

[*Note:* Existing information about Abu Bakr Muhammad 'Abdallah' Ayyash **al-Ḥaṣṣâr** is limited; a few remarks are contained in **http://en.wikipedia.org/wiki/Abu_Bakr_al-Hassar** and **Al Hassar and fractions.pptx.]**

Several other standard symbols date back to the 17th century and will be mentioned later in the notes when we cover that period (Unit 11). The following online sites contain more comprehensive information about the development of mathematical symbols:

http://www.unisanet.unisa.edu.au/07305/symbols.htm

http://jeff560.tripod.com/mathsym.html

A few additional comments about Chuquet and his notational innovations deserve to be added. Namely, he introduced symbolism for the 0^{th} power and also allowed the use of negative numbers as exponents. He also expanded upon the terminology for large numbers which apparently existed at the time (million, billion, trillion) to the next level with the concepts of quadrillion, quintillion, and so on; this nomenclature is often known as the ***Chuquet system.*** It differs from current American usage in that each number in Chuquet's system was one million times the previous one (as in some European languages for which a billion equals a million million, rather than a thousand million as in the U.S.). Chuquet's known work, *Triparty en la science des nombres,* was the first algebra book in French. In many respects (for example, the exponential notation) Chuquet was far ahead of his time, but his insights went largely unrecognized until the manuscript was rediscovered in the 1870s.

Definitive mathematical references in print

One of the most prominent Italian teacher – mathematicians from the late 15th and early 16th centuries was L. Pacioli (1445 – 1517), who in 1494 published a summary of known mathematics at the time in his book, *Summa de arithmetica, geometria, proportioni et proportionalita* (1494), designed for schools in Northern Italy. This encyclopedic book had significant defects, but it circulated widely and provided a basis for the major advances in mathematics during the early 15th century, and it is noteworthy for the degree to which it introduced shorthand notation for standard mathematical objects and operations; there are a few comments about this book on pages 316 – 317 of Burton. Pacioli's book also had a particularly strong impact because it contained a lengthy, detailed summary of double entry bookkeeping which was regarded as definitive for a long time.

Another financially related item in Pacioli's book is the often used **Rule of 72** for estimating the amount of time in which an investment's value will double; more precisely, if the annual percentage rate of interest is R per cent and the interest is compounded at least annually, then it will take about $72/R$ years for the investment's double in value. Pacioli does not explain or derive this rule, but the underlying mathematics is described in **http://math.ucr.edu/~res/math153-2019/compounding.pdf**; the significance of 72 is that it is divisible by many small integers and is close to 100 times the natural logarithm of 2. The article **http://en.wikipedia.org/wiki/Rule_of_72** contains additional information.

The *Summa* was only one of Pacioli's written works, which include an unpublished manuscript on recreational problems, geometric problems and proverbs. The latter frequently refers to Leonardo da Vinci, a close lifelong associate of Pacioli who worked with him on this venture.

Many books on such mathematical topics were published beginning in the late 15th century and continuing through the 16th century. In the German school, perhaps the best known author was A. Riese (1492 – 1559), whose book *Coss* was published in 1525; this book uses decimal fractions and modern notation for roots. One indication of the book's impact is that the phrase "according to Adam Riese" is still a widely used German idiomatic expression for absolute mathematical reliability. Other examples include books by Stifel, Rudolff, Widman and J. Scheubel (1494 – 1570) in German, E. de la Roche (c. 1470 – c. 1530) in French, and Recorde, C. Turnstall (1474 – 1559) and J. Dee (1527 – 1608) in English. Since Scheubel is not in the MacTutor list, here is an online reference for a biographical sketch (it is translatable using free online software):

http://de.wikipedia.org/wiki/Johann_Scheubel

Mathematical mapmaking

[***Note***: This material is not covered in Burton.]

Strictly speaking, this material belongs in a discussion of 16th century mathematics, but we are including with 15th century advances because of its close ties to trigonometry.

With the increase of long distance maritime activity throughout the 15th century, there was a correspondingly greater need for accurate and user – friendly maps. One of the first persons to study such problems in depth was P. Nunes (noo – nesh, 1502 – 1578).

He worked in many areas of science, with his main contributions to the theory of mapmaking in his 1537 book, *Tratado da sphera* (Treatise on the sphere), and he is known for discovering a spherical curve called the ***rhumb line*** or ***loxodrome***, a spiral curve ending at one of the poles which moves in a steady directional course (for example, constantly in the north – northwest direction). Some aspects of Nunes' work were ahead of their time; further information on these and other contributions are given in the online article **http://en.wikipedia.org/wiki/Pedro_Nunes**. Here is a reference for the loxodrome curve; further information also appears in supplementary file **(8.A):**

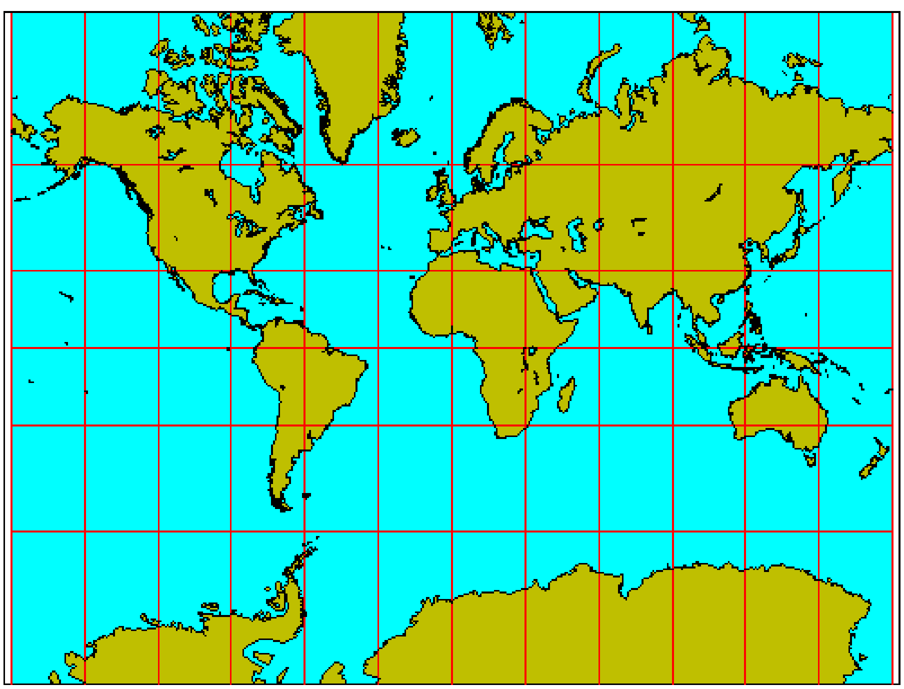
http://en.wikipedia.org/wiki/Loxodrome

Probably the best known mapmaker from this period was Gerardus Mercator (1512 – 1592; this name is latinized from Gheert Cremer) after whom the familiar ***Mercator map projection*** is named. This projection succeeded in solving several key problems that Nunes studied without success.

Here are several important features of the Mercator projection:

1. Latitudes and longitudes are represented by horizontal and vertical lines.
2. Loxodromes are represented by oblique lines.
3. The map projection preserves the angles at which curves intersect.
4. Regions far from the equator are severely distorted.
5. To compensate for incorrect spacing between meridians, the spacing of latitudes towards the poles is increased.
6. The projection is good for determining the compass direction between two points, but not at all accurate for estimating distances traveled or comparing areas of land masses (despite what the map below suggests, Australia is more than 3½ times as large of Greenland!).

This is a typical example of a Mercator projection map:



(***Source:*** **http://www.uwsp.edu/geo/projects/geoweb/participants/Dutch/FieldMethods/UTMSystem.htm**)

There is a detailed discussion of this map projection in the online file

http://en.wikipedia.org/wiki/Mercator_projection

which gives the following formulas for finding the $x-$ and $y-$ coordinates in the Mercator projection plane from the latitude φ and longitude λ (where λ_0 is the longitude at the center of map):

$$\begin{aligned} x &= \lambda - \lambda_0 \\ y &= \ln\left(\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\right) \\ &= \frac{1}{2}\ln\left(\frac{1+\sin(\varphi)}{1-\sin(\varphi)}\right) \\ &= \sinh^{-1}\left(\tan(\varphi)\right) \\ &= \tanh^{-1}\left(\sin(\varphi)\right) \\ &= \ln\left(\tan(\varphi) + \sec(\varphi)\right). \end{aligned}$$

The finishing touches to the mathematics of the Mercator projection were completed by E. Wright (*c.* 1558 – 1615), T. Harriott (1560 – 1621), and H. Bond (*c.* 1600 – 1678); we shall say more about Harriott's highly significant contributions later. Numerous other map projections have appeared since Mercator's, with wide ranges of objectives such as minimizing area distortions and providing convenient frameworks for finding the shortest paths (***great circles***) from one point on the earth to another (recall that a ***great circle*** on a sphere is a circle whose center is also the center of the sphere).

(***Source:*** **http://earthquake.usgs.gov/learn/glossary/?term=great%20circle**)

For example, polar projection maps give much better ways to approximate great circles than maps like the Mercator projection.

(Source: https://www.winwaed.com/blog/2010/01/11/polar-maps-and-projections-part-1-overview/)

Final remarks on mapmaking

In the 18th century J. H. Lambert (1728 – 1777) studied problems of mapmaking in considerable detail. Most significantly, his book ***Notes and Comments on the Composition of Terrestrial and Celestial Maps*** introduced several new or improved map projections which have been used extensively ever since.

Everyday experience suggests that we cannot flatten out a piece of a sphere so that it fits onto a plane without any distortion of distances or angle measurements, and one way of reaching this conclusion arises from the fact that the angle sum of a spherical triangle is always greater than 180 degrees (in contrast, one obviously can do this with a piece of a cylinder). However, a full proof of this was not given until the 19th century work of C. F. Gauss on the geometry of curved surfaces.

Many elementary aspects of mapmaking theory are covered from a mathematical viewpoint in the following undergraduate level book:

> **T. G. Freeman, *Portraits of the Earth: A Mathematician Looks at Maps*.**
> American Mathematical Society, Providence RI, 2002.

Addenda to this unit

There are four separate items. The first document (**8.A**) discusses loxodromic curves in more detail, the second (**8.B**) sketches a proof that there is no regular decomposition of $\mathbf{3}$ – space with regular tetrahedra, the third (**8.C**) lists some websites with interactive $\mathbf{3}$ – dimensional graphics , and the fourth (**8.D**) contains more information on the concept of vanishing points in perspective drawing.