8.D. Vanishing points in perspective drawing

In the main notes we mentioned the *vanishing point property* in the theory of perspective drawing:

THEOREM. *Let $\mathbf{E}$ be a point in space, let P be a plane not containing $\mathbf{E}$, and let $\mathcal{H}$ be the half-space of all points such that $\mathbf{E}$ and $\mathcal{H}$ lies on opposite sides of P. Define the perspective projection φ on $\mathcal{H}$ such that φ sends a point $\mathbf{x}$ to the unique point at which the line $\mathbf{Ex}$ meets the plane P. For each line L which meets P in a single point (so L is neither parallel to nor contained in P) there is a point $\mathbf{v}(L)$ on the image line containing $\varphi[L]$ such that $\mathbf{v}(L) = \mathbf{v}(L')$ for every line L' which is parallel to L.*

The drawings on page 6 of `history08.pdf` illustrate this phenomenon. The *vanishing point* $\mathbf{v}(L)$ is the limiting position of the points $\varphi[\mathbf{x}_t]$ on the line containing $\varphi[L]$; recall from `history08.pdf` (see p. 5) that the image $\varphi[L]$ is contained in some line (but usually it is a proper subset of that line).

We shall prove a slightly simplified version of this result using coordinate geometry. The ideas in this proof can be extended to prove the general theorem as stated above, but the algebra gets more complicated.

Proof. For the sake of definiteness we shall take the image plane P to be the xz-plane, which is defined by the equation $y = 0$, and we shall take the eye position $\mathbf{E}$ to have coordinates $(0, -1, 1)$. Also, we shall take L to be a line which passes through the point $\mathbf{c} = (q, 0, r)$ of the plane P, with direction given by the vector $\mathbf{w} = (a, 1, b)$ (in other words, L is neither parallel to a line in P nor contained in P). Finally, note that $\mathcal{H}$ is the set of all points with positive second coordinates.

By construction, the points on $L \cap \mathcal{H}$ have the form $\mathbf{x}_t = \mathbf{c} + t(a, 1, b)$ where $t > 0$. The coordinates of the image point $\varphi(\mathbf{x}_t)$ are determined by two conditions.

(1) Since the image point lies on the line joining $\mathbf{E}$ to $\mathbf{x}_t$, we know that $\varphi(\mathbf{x}_t)$ is equal to $(1-u)\mathbf{E} + u\,\mathbf{x}_t$ for some real number u (which must be strictly between 0 and 1).

(2) Since the image point lies on the xz-plane, the second coordinate of the expression in (1) must be zero.

If we combine the preceding observations, we obtain the following equations:

$$\varphi(\mathbf{x}_t) \quad = \quad (0, u-1, 1-u) \; + \; u(q + ta, t, r + tb)\ , \qquad u - 1 + tu \quad = \quad 0$$

The second equation implies that $u = 1/(1+t)$, and it follows that $1-u$ must be equal to $t/(1+t)$, which is also equal to tu. Notice that u is always between 0 and 1 because t is positive. Using this we can write the left hand side entirely in terms of the remaining data as follows:

$$\varphi(\mathbf{x}_t) \quad = \quad \left(\frac{q + ta}{1+t},\ 0,\ \frac{r + tb}{1+t} \right)$$

It is now straightforward to check that the limit of the right hand side as $t \to \infty$ is equal to $(a, 0, b)$. Therefore the limit only depends upon the direction vector $(a, 1, b)$ for the original line L, which means that the limit will be the same for all lines parallel to L.■