Sample Cubic Equation (From the 2005 final)

Use a change of variables $y = x + a$ to rewrite $x^3 + 6x^2 + 11x + 6 = 0$ as $y^3 + py + q = 0$.

<u>General approach</u> $0 = x^3 + bx^2 + cx + d \Rightarrow$ take $y = x + \frac{1}{3}b$, so $x = y - \frac{1}{3}b$ and
$0 = (y - \frac{1}{3}b)^3 + b(y - \frac{1}{3}b)^2 + c(y - \frac{1}{3}b) + d$ can be re written as $y^3 + py + q$. — In our example $b = 6$, so $x = y - 2$ and hence we have

$$0 = (y-2)^3 + 6(y-2)^2 + 11(y-2) + 6 =$$
$$(y^3 - 6y^2 + 12y - 8) + (6y^2 - 24y + 24) + (11y - 22) + 6 =$$
$$\ldots = \boxed{y^3 - y}$$

<u>Now find the roots of the original polynomial.</u>

The roots of the new polynomial are $y = 0, \pm 1$, since $y^3 - y = y(y-1)(y+1)$. Since $x = y - 2$, it follows that

$$x = \begin{Bmatrix} 1 - 2 \\ 0 - 2 \\ -1 - 2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ -2 \\ -3 \end{Bmatrix}.$$