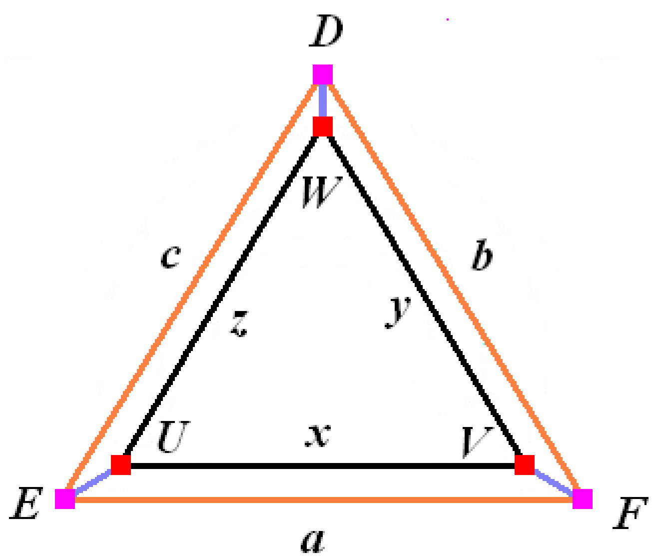
The Delta Resistance Problem

The following problem was part of a physics laboratory exercise in an undergraduate physics course at the University of Chicago during the Winter 1963 Quarter. We are given an electrical network as shown below, in which we can measure resistance from D to E, from E to F, and from D to F. The values across these resistances will be called c, a, and b respectively. The wiring is given by the short blue and the black lines, and the orange line indicates extra pieces across which the resistances can be measured separately. In particular, if we complete the circuit with the orange line from D to E, then the total resistance c from D to E is given by two parallel circuits, one of which has a resistance of z and the other of which as a total resistance of $x + y$ (since the resistances x and y are in series). Analogous statements hold for the resistances between the other pairs of points with suitable permutations of the variables. The problem is to compute the values for x, y, and z (which for some unknown reason **cannot** be measured directly) from the values for a, b, and c (which **can** be measured directly).



If we apply the formulas for resistances in series and parallel, we obtain the following system of three equations in x, y, z in terms of a, b, and c.

$$\frac{1}{a} = \frac{1}{x} + \frac{1}{y+z}$$

$$\frac{1}{b} = \frac{1}{y} + \frac{1}{x+z}$$

$$\frac{1}{c} = \frac{1}{z} + \frac{1}{x+y}$$

It seems clear that there should be a unique solution for x, y, z in terms of $a, b, c,$ but of course we want to have some formula for this solution. One of the first things that one might think of doing is to clear this system of fractions, and if we do so we obtain the following equivalent system:

$$\begin{aligned} xy + xz &= a(x + y + z) \\ xy + yz &= b(x + y + z) \\ xz + yz &= c(x + y + z) \end{aligned}$$

How can we solve this efficiently to find neat formulas for the unknowns, especially if we know that all the resistances $x, y, z, a, b,$ and c are positive? Here is a fairly simple method:

If we let $S = x + y + z$, then we can rewrite the system above as a system of three linear equations in $xy, yz,$ and xz in which the right hand sides are positive multiples of S. We can solve this system for x, y, z in terms of $a, b, c,$ and $S,$ and if we do so we obtain the following set of equations:

$$xy = \frac{(c-b-a)(x+y+z)}{-2} = \frac{(a+b-c)(x+y+z)}{2}$$

$$yz = \frac{(b+c-a)(x+y+z)}{2} \qquad xz = \frac{(a+c-b)(x+y+z)}{2}$$

The important point is that the equations express $xy, yz,$ and xz as nonzero expressions in $a, b,$ and c times the sum $S = x + y + z$. Let's assume that none of a, b, c is equal to the sum of the other two, which will be the case if $x, y, z, a, b,$ and c are all positive. In this case we can divide each of these equations by either of the remaining ones, and if we do so we obtain the following:

$$\frac{y}{x} = \frac{yz}{xz} = \frac{b+c-a}{a+c-b} = p\ , \quad \frac{z}{x} = \frac{yz}{xy} = \frac{b+c-a}{b+a-c} = q$$

Note that the quantities p and q are expressed in terms of $a, b,$ and $c,$ and it will be convenient to include these in our list of known quantities. The two equations above yield expressions for y and z in terms of x and the known variables, so it will be enough to solve for x in terms of these known variables. If we now substitute the expressions for y and z in terms of x and the known variables into the first of the resistance equation, we obtain the following:

$$\frac{1}{a} = \frac{1}{x} + \frac{1}{px+qx} = \frac{1}{x}\left(1 + \frac{1}{p+q}\right) = \frac{1}{x}\left(\frac{p+q+1}{p+q}\right)$$

We can now solve for x in terms of $a, b,$ and q:

$$x = a \cdot \left(\frac{p+q+1}{p+q} \right)$$

Once we know x we can find the remaining unknowns from the previously derived pair of equations $y = px$ and $z = qx$. These express the unknown resistances x, y, and z as rational functions of the known resistances a, b, and c. To be certain these formulas yield legitimate expressions, we need to check that $p + q$ is nonzero. From the formulas for p and q we conclude that $p + q$ equals zero if and only if $a + c - b = c - a - b$, which translates to the condition $a = 0$. However, we are assuming the measured resistances are all nonzero, so we do not have to worry about this possibility.

<u>Note.</u> One can avoid many problems involving zero denominators in this derivation by working with indeterminates in a suitable polynomial ring. The unknowns can then be found by substituting in specific values in place of the indeterminates and recognizing that the physics of the problem implies that the specific values are always nonzero.