9. Mathematics in the sixteenth century

(Burton, **7.2 – 7.4**, **8.1**)

The 16$^{\text{th}}$ century saw several important developments, some of which pointed to definitive resolutions of themes from earlier centuries and others of which reflected the movement of the subject into entirely new directions. For example, there was major progress in developing a common, user – friendly symbolic language for mathematics which was essentially completed during the next century, settling an issue that many mathematicians had struggled with for 1400 years. Perhaps the most celebrated breakthrough in a new direction was the derivation of formulas for solving third and fourth degree polynomial equations in one unknown. The quadratic formula had already been known for about 3500 years, and even in 1494 Pacioli had stated that a formula for the third degree equation seemed like an unattainable goal.

The cubic and quartic formulas

We have already noted that accurate versions of the quadratic formula were known to the Babylonians by 2000 B.C.E., and that various mathematicians had devised methods for finding solutions cubic equations. However, for third and higher degree equations there was nothing comparable to the quadratic formula for finding solutions, and even at the end of the 15$^{\text{th}}$ century there were strong doubts that such formulas existed. During the early and middle 16$^{\text{th}}$ centuries mathematicians discovered formulas for the roots of cubic and quartic (fourth degree) polynomials in terms of the polynomial's coefficients. The colorful details of the discovery and publication of the cubic formula are recounted in Section 7.2 of Burton, particularly on pages 316 – 320. The mathematical aspects can be summarized by noting that the original formula was discovered in one basic case but not published by S. del Ferro (1465 – 1526), rediscovered independently and extended to other cases by N. Fontana, who is better known as Tartaglia (1500 – 1557), and published by G. Cardano (1501 – 1576); the latter knew about Tartaglia's work and did not have his permission to publish it, but as noted in Katz there may be two sides to this story. As noted in Burton, although Cardano (also called Cardan) had a well deserved reputation for being very unscrupulous, he also made important contributions to mathematics that cannot be discounted. His most important book, ***Ars Magna***, was devoted to algebra as it was known at the time and included a great deal of important material that was unquestionably his own.

The main idea of the derivation of the cubic formula is to make a clever change of variables which takes a cubic equation in some variable x and transforms it into a quadratic equation in some related variable z^3. One then solves for z^3 by the quadratic formula and substitutes back to obtain the desired formula for x. We shall give a derivation of the basic formula using a simplification introduced by F. Viète (also called Vieta, 1540 – 1603), whose main work will be discussed in the final part of this unit.

First of all, since the coefficient of x^3 is nonzero and multiplying a polynomial by a nonzero constant does not change the roots, we may as well assume that the coefficient of this leading term is equal to 1. Thus we have an equation of the form

$$z^3 + a_2 z^2 + a_1 z + a_0 \quad = \quad 0$$

and if we make the change of variables $z = x - (a_2/3)$ we eliminate the quadratic term and obtain

$$x^3 + px = q$$

(see **http://math.ucr.edu/~res/math153/cubic-example.pdf** for an example of how such a change of variables can simplify things). The next step is to make a second change of variables

$$x = w - \frac{p}{3w}$$

which leads to the equation

$$w^3 - \frac{p^3}{27w^3} - q = 0$$

and we obtain the following polynomial equation if we multiply both sides by w^3:

$$w^6 - qw^3 - \frac{p^3}{27} = 0$$

We can now solve this for w^3 using the Quadratic Formula and extract cube roots to find w itself. After doing so we may use the resulting values for w to find x and z in succession. The Tartaglia – Cardano form of the solution is then expressed in terms of x as follows:

$$\sqrt[3]{\sqrt{(p/3)^3 + (q/2)^2} + (q/2)} - \sqrt[3]{\sqrt{(p/3)^3 + (q/2)^2} - (q/2)}$$

For the sake of completeness we shall present the explicit formula for the general cubic equation $ax^3 + bx^2 + cx + d = 0$ in terms of the coefficients:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}.$$

Clearly this formula is much more complicated than the quadratic formula, and in contrast to the latter ***it is not particularly useful for computing the roots of an arbitrary cubic polynomial.*** In fact, as Cardano noted, if one considers the equation

$$x^3 = 15x + 4$$

for which **4** is easily checked to be a root (and the remaining two roots are real), then the formula yields the following surprising expression:

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

There are many cubic polynomials for which one obtains such expressions as roots, even in cases where all three roots are real; such cases were said to be ***irreducible.*** Cardano did not know how to interpret the formula for the irreducible polynomial considered above, but subsequently R. Bombelli (1526 – 1572) showed how to do so using complex numbers (we should note that Cardano mentioned complex numbers just once in his book, ***Ars Magna***). In particular, Bombelli used complex numbers to explain why the expression on the right side of the formula is equal to 4 (see pages 325 – 326 of Burton for all the details). This insight was an important step towards the ultimate acceptance of complex numbers by mathematicians by the beginning of the 19th century. Further details on these points appear in Section 7.3 of Burton. The next section in Burton (7.4, pages 328 – 333) discusses the analogous formula for fourth degree equations due to L. Ferrari (1522 – 1565).

Notes on complex arithmetic and its applications. There are discussions of the application of complex numbers to the theory of alternating current electrical circuits in supplement **(9.A)** and the following two files:

http://math.ucr.edu/~res/math153-2019/impedance.pdf

http://math.ucr.edu/~res/math153-2019/impedance2.pdf

Using the cubic formula. A few simple and clearly presented examples of computations with the cubic formula appear in pages 3 – 5 the following online document:

https://www.researchgate.net/publication/228392121_An_easy_look_at_the_cubic_formula/download

Significance of the cubic and quartic formula discoveries. Although the cubic formula is generally not all that helpful for finding roots of polynomials and numerical methods are often indispensable for describing such roots, this formula still ranks as one of the most important discoveries in mathematics because of its influence on mathematical thinking. First of all, it was a breakthrough that went far beyond anything that ancient mathematicians had done and indicated that mathematics was poised to answer new sorts of questions. Also, the solution emphasized the need to work with negative numbers, and even complex numbers, in some problems that only seemed to involve positive real numbers. As time progressed, numerous other examples of this sort arose. Finally, since the cosine of $3x$ is equal to a cubic polynomial in $\cos x$, the cubic formula led mathematicians to view the classical Greek trisection problem in terms of algebra, and the success in finding a formula for third and fourth degree equations led to extensive studies of fifth degree equations during the next 250 years.

Nonexistence of a quintic formula. Late in the 18th century P. Ruffini (1765 – 1802) described a method for showing that one could ***not*** find a ***quintic formula***; *i.e.*, an expression giving the roots of a general fifth degree equation in one variable in terms of addition, subtraction, multiplication, division and extraction of n^{th} roots, where $n = 2, 3, 4$ or 5. Several years later, N. H. Abel (1802 – 1829) gave a more insightful and rigorous argument, and the same ideas show that there also cannot be a similar sort of formula for n^{th} degree equations for any larger values of n. In fact, it is possible to construct explicit, fairly simple quintic polynomials whose roots cannot be expressed in terms of the coefficients using the standard four basic arithmetic operations (addition, subtraction, multiplication, division) and taking n^{th} roots. One such example is the polynomial $p(x) = 3x^5 - 15x + 5$. There is a detailed discussion of this example on pages 556 – 558 of the following book:

J. Gallian. *Contemporary Abstract Algebra* (7th Ed.).
Brooks/Cole, Belmont CA, 2009.

Actually, it is possible to write down a quintic (fifth degree) formula if one introduces just one more operation; namely, evaluation of a number using a function $g(x)$ that is inverse to the polynomial $p(x) = x^5 + x$. Over the real numbers, one can check that the function $p(x)$ defines a $1-1$ correspondence of the real line with itself because its derivative is always positive (hence it is strictly increasing), and furthermore the limits of $p(x)$ as x tends to $\pm\infty$ are $\pm\infty$ (respectively). These observations imply that ***for each real number*** y ***one can find a unique real number*** x ***such that*** $y = x^5 + x$, and therefore an inverse function $g(x)$ satisfying $x = g(x)^5 + g(x)$ actually exists.

Here are some online references for quintic equations and some of the material discussed above. The second is an electronic version of a large poster covering nearly the entire history of research on quintic equations.

http://mathworld.wolfram.com/QuinticEquation.html

http://library.wolfram.com/examples/quintic/

The emergence of symbolic and decimal notation

By the middle of the 16th century various mathematical symbols and abbreviations were widely used, but many authors — for example, Cardano — still formulated much of their work rhetorically. However, towards the end of the century the situation changed rapidly, and by the beginning of the 17th century early versions of modern symbolic notation had become fairly well established. The crucial step was to move beyond simple abbreviations for mathematical terms and concepts and to introduce symbols which might have no evident ties to the concepts or objects they represent.

We have already noted that the path to developing symbolic notation for mathematics was long (nearly 1400 years from Diophantus' initial efforts), uneven, and often tentative. It seems likely that there were several reasons why the transition to symbolic notation took so long. We have already noted that many writers used versions of syncopated notation during the late Middle Ages, but different authors used different systems, and uniform usage did not really begin to evolve until after the invention of the printing press. Also, the impact of Diophantus' work on Renaissance mathematics was limited until the appearance of Latin translations near the end of the 16th century. Regiomontanus had lamented the lack of such a translation a century earlier and had apparently planned to translate the manuscripts himself, but this was never done and even leading 16th century algebraists like Tartaglia and Cardano do not appear to have known anything about Diophantus' writings. It is tempting to speculate what might have happened if Diophantus' writings had been more widely known in the early part of the 16th century. Other potential reasons for the delay are described in the following comments from an apparently defunct website (**www.brynmawr.edu/math/people/anmyers/295/Renaissance.pdf**):

> Until this point in the history of mathematics, algebraic problems were either presented verbally or described using a combination of words and abbreviations (in the manner of Diophantus [and some Indian mathematicians]). The fact that it took so long to develop algebraic notation is a testament to its conceptual difficulty. Many students have a hard time making sense of the symbols used in algebra. It took mathematicians centuries (millennia, actually) of working verbally with algebra until the concepts were familiar enough to be able to meaningfully describe with abstract labels.

Another way to appreciate the difficulty in finding adequate symbolism is to imagine the challenges that early civilizations must have faced in developing written languages from spoken ones. Mathematicians faced a task which was probably similar in at least some respects.

<u>The work of François Viète</u>. The most influential and decisive contributor to the new symbolic notation was F. Viète (1540 – 1603, also known as Vieta or Vièta), whom we have already mentioned in connection with the cubic formula. He brought together many scattered ideas that were in circulation and added some important ones of his own in his work, *An Introduction to the Art of Analysis (In artem analyticam isagoge)*. Here are some particularly significant new ideas he contributed:

1. He used letters in equations to denote both known and unknown quantities.
2. He consolidated different types of equations by using both addition and subtraction (thus allowing unified approaches to families of equations like $x^2 = bx + c$, $x^2 + c = bx$ and $x^2 + bx = c$ which had been treated separately since the time of Al-Khwarizmi).
3. His use of symbolism was not casual but systematic.

Very similar ideas were advanced by T. Harriot (1560 – 1621), whose works were not published until after his death. There are several unanswered questions about the extent to which Harriot and Viète were acquainted with each other's work or influenced each other.

Further discussion of Viète's notational conventions and R. Descartes' adjustments to them appear on pages 347 – 348 of Burton. Perhaps the most noteworthy thing to repeat here is that Viète used vowels to indicate unknowns and consonants to indicate known quantities, while Descartes used letters at the end of the alphabet to indicate unknowns and letters at the beginning of the alphabet to indicate known quantities.

In our discussion of 17th century mathematics we shall describe how Viète's advances in symbolic notation and subsequent improvements led to many of the symbolic conventions that have been use ever since the first half of the 17th century, concluding the transition from words to symbols which began with the work of Diophantus about 1400 years earlier. It is somewhat curious that uniform symbolic mathematical notation developed much later than similar notation for alchemy, astrology and music, but we shall not try to speculate about possible reasons.

Viète's other mathematical work included results on the theory of (roots of polynomial) equations, including the basic relationships between the coefficients of a polynomial and its roots. He also made highly significant contributions to trigonometry and the relationship between trigonometric identities and finding roots of polynomials; for example, his results yield solutions to the irreducible cubic equations that Cardano had attempted to understand, he gave a solution of the general cubic that requires the extraction of only a single cube root, and he showed that both the classical angle trisection and cube duplication problems depend upon solutions to cubic equations; see (**9.C**) for details. The online site

http://math.berkeley.edu/~robin/Viete/construction.html

describes his construction of a regular 7 – sided polygon using complex roots of cubic polynomials.

The following anecdote regarding Viète illustrates his use of trigonometry to find roots of polynomials. In 1593, A. van Roomen (1561 – 1615) issued an open challenge to solve the following polynomial equation of the 45th degree:

$$x^{45} - 45x^{43} + 945x^{41} + \ldots - 3795x^{3} + 45x = K.$$

Viète noticed that this equation arises when one expresses $K = \sin 45y$ in terms of $x = 2\sin y$, and he quickly found about 20 roots to this equation (one reason he did not find more is that he only considered positive roots — complete and universal acceptance of negative numbers would finally take place by the middle of the 17th century, about 1000 years after Brahmagupta used them freely in his writings; resistance to the use of negative numbers is even seen in some books written near the end of the 18th century).

Decimal expansions. The end of the 16th century also saw the adoption of the decimal fraction numbering system essentially as we know it. In the discussion of Fibonacci (as given in Burton) it is noted that he still used Babylonian sexagesimal notation for fractions even though he used Hindu – Arabic numerals for whole numbers, and in fact for some time this was common practice. As noted in Unit 6, some Arabic mathematicians had discussed decimal fractions at length, most notably Al-Kashi, and Chinese mathematicians had also used the decimal concept fairly systematically in their algorithms, but it was not until the appearance of *La Theinde* (also known as *Disme, The Arts of Tenths or Decimal Arithmetike*) by Simon Stevin (1548 – 1620) that the use of decimal fractions became widespread in Europe for general purposes (see pages 324 – 325 of Burton for additional information); not surprisingly, the decimal approach in Stevin's work anticipated the innovative adoption of decimal coinage in the United States near the end of the 18th century (in fact, the first U.S. coins, which were made of silver with a face value of 5 cents, are inscribed "half disme") and the momentous French development of the metric system for weights and measures shortly afterwards (but it should be noted that such systems had been proposed beginning the second half of the 17th century).

(***Source:*** **http://www.coin-collecting-guide-for-beginners.com/half-dimes.html**)

Further work on developing the modern notion of decimal expansions was completed by C. Clavius (1538 – 1612) and J. Napier (1550 – 1617; his work on logarithms will be discussed in Unit 11).

Note: In view of the early adoption of decimal coinage, it is somewhat ironic that the United States is one of the very few countries in which the metric system is not universally employed (Liberia and Myanmar are apparently the only others; for more information see the article **https://en.wikipedia.org/wiki/Metric_system**), but its use in the U.S. has been legal ever since the middle of the 19th century.

Another work by Stevin (*L'arithmétique*) marked an important conceptual milestone in the development of a unified approach to rational and irrational numbers. We have already noted that the existence of irrational numbers was an issue that Greek mathematicians found awkward to handle; two illustrations of this are **(1)** the Greek distinction between rational numbers and geometrical magnitudes, **(2)** the extremely lengthy and somewhat rambling discussion of irrationals in Book **X** of Euclid's ***Elements*** (comprising nearly ***one fourth*** of the entire ***Elements***). As previously noted, Indian and Arabic mathematics were far more willing to consider irrational and rational quantities together, and in practice European mathematicians had largely freed themselves from the Greek constraints by the end of the 16th century; in effect, Stevin's work in *La Theinde* and *L'arithmétique* marked the final acceptance and legitimization of this viewpoint. Stevin's acceptance of negative numbers was also an important step to the free use of negative numbers by the end of the 17th century; on the other hand, he had reservations about imaginary numbers.

Although Stevin might not be as well known as other major figures from the Renaissance (for example, Leonardo da Vinci), he also made numerous important contributions to a wide range of subjects in the sciences and engineering. The articles

http://en.wikipedia.org/wiki/Simon_Stevin

http://www.nndb.com/people/895/000096607/

summarize much of this work.

Addenda to this unit

There are three separate items. The first document (**9.A**) explains why $\cos(360°/7)$ is a root of a cubic polynomial with integral coefficients, the second (**9.B**) illustrates how complex arithmetic can be used to work with Ohm's Law for alternating current (AC) circuits, and the second (**9.C**) discusses the neusis constructions considered by Viète and their relation to finding the roots of cubic and quartic polynomials. In addition, there are two other files (**impedance.pdf** and **impedance2.pdf**) which deal with complex numbers and AC circuits, and there are also two additional files (**neusis-geometry.pdf** and **neusis-algebra.pdf**) which deal with neusis constructions.

Appendix: *The mathematics of decimal expansions*

There is an elementary but detailed mathematical discussion of decimal expansions (and also expansions in bases other than $\mathbf{10}$) on pages 97 – 110 of the following online notes:

http://math.ucr.edu/~res/math144-2017/setsnotes5.pdf

In particular, on pages 99 – 100 and 105 – 106 of that document there is a statement and proof for a standard characterization of decimal expansions for rational numbers (see Theorems $\mathbf{V}$.5.13 and $\mathbf{V}$.5.14):

Eventual Periodicity Property. *Suppose that* r *is a rational number such that* $0 < r < 1$, *and let*

$$r = b_1 \cdot 10^{-1} + b_2 \cdot 10^{-2} + \ldots + b_k \cdot 10^{-k} + \ldots$$

be a decimal expansion (so that each b_k *is an integer satisfying* $0 \leq b_k \leq 9$*). Then* r *is a rational number if and only if the sequence* $\{b_k\}$ *is* ***eventually periodic***; *i.e., there are positive integers* M *and* Q *such that* $b_k = b_{k+Q}$ *for all* $k > M$.

One direction of this result (***eventually periodic decimal expansions always determine rational numbers***) is a fairly simple consequence of the formula for summing a geometric series; for example, if each of the terms b_k is equal to 1 then r is equal to $1/9$ (verify this!). In fact, for this example the sequence $\{b_k\}$ is periodic. A simple example for which $\{b_k\}$ is **NOT** periodic is $1/6$; its decimal expansion is $0.16666\ldots$, so that $b_1 = 1$ but $b_k = 6$ for all $k > 1$.

Still further information is summarized in the following online documents:

http://mathworld.wolfram.com/DecimalExpansion.html

http://www.lrz.de/~hr/numb/period.html

http://www.geom.uiuc.edu/~rminer/1over89/