Neusis solutions to classical construction problems

This material is taken from an older set of notes. There are some difference in notation between this document and the course notes. Most notably, **X – Y – Z** means that **Y** is between **X** and **Z**, while **|AB|** denotes the distance between A and **B**, and (if **X** does not lie on the line **AB**) the **X** – ***side of*** **AB** consists of all points in the plane **XAB** which are on the same side of **AB** as **X.**

Note on Step 3, page 19. A reference to other parts of the old notes is given for a proof of the claim. In the course notes, the appropriate reference is Lemma **III.4.7.**

Trisections and cube duplications using other instruments

Although trisections and cube duplications cannot be done by marked straightedge and compass, these constructions can be performed using other instruments and curves other than lines and circles. The first results were already known to Greek geometers. There was a revival of interest from the sixteenth to the nineteenth century. Many of these are summarized in [Eves, 1969], Problem Studies for Chapter 4.

We shall show that both trisection and cube duplication become possible if we are allowed to place two marks on the straightedge; this is about the simplest elaboration one can imagine, and it certainly applies to ordinary rulers, which of course have dozens of marks on them (one must pick two of the marks and ignore the others).

Notation. The distance between the two marks on the straightedge will be denoted by d.

Angle Trisection

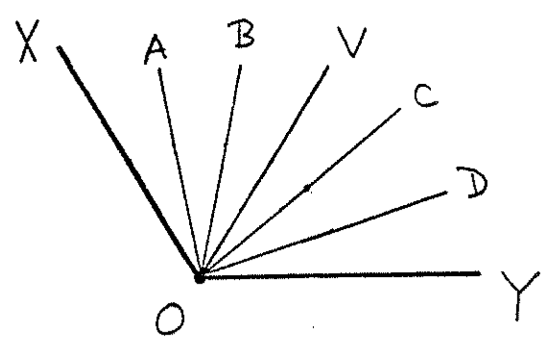
We begin with a reduction to the following problem:

CONSTRUCTION 1. Let $\angle XOY$ with $|\angle XOY| < 90$ be given. Using marked straightedge and compass, it is possible to find $Z \in \text{INT } \angle XOY$ so that $|\angle ZOY| = |\angle XOY|/3$.

Why Construction 1 implies arbitrary angle trisection. If $|\angle XOY| < 90$, we can bisect $\angle ZOX$ by (unmarked) straightedge and compass to get $[OW$ such that

$$|\angle XOW| = |\angle WOZ| = |\angle ZOY| = |\angle XOY|/3.$$

If $|\angle XOY| = 90$, we can construct $Z, W \in \text{INT } \angle XOY$ such that $|\angle ZOY| = |\angle WOX| = |\angle ZOW| = 30$ directly. Finally, if $|\angle XOY| > 90$, first bisect $\angle XOY$ using $[OV$ by (unmarked) straightedge and compass.

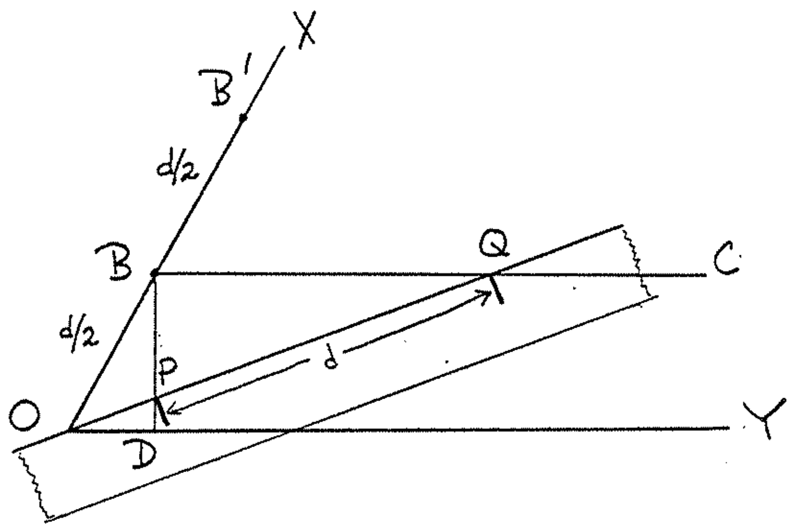


Now trisect each of the acute angles $\angle XOV$ and $\angle YOV$ by $[OA$, $[OB$ and $[OC$, $[OD$ (i.e., one has $|\angle VOC| = |\angle VOB| = |\angle XOY|/6$ and $C \in \text{INT } \angle VOY$, $B \in \text{INT } \angle VOX$). It then follows that $B, C \in \text{INT } \angle XOY$ and

$$|\angle XOB| = |\angle BOC| = |\angle COY| = |\angle XOY|/3.$$

(Details at many places are left to the reader.)

PROCEDURE FOR CONSTRUCTION 1. *Step 1:* Using the marked straightedge, locate $B' \in (OX$ so that $|OB'| = d$.



Step 2: Using (unmarked) straightedge and compass, find the midpoint B of $[OB']$.

Step 3: Using (unmarked) straightedge and compass, drop a perpendicular from B to OY with foot D. CLAIM: D lies on (OY {use the results of Section 14.1 in this syllabus}.

Step 4: Using (unmarked) straightedge and compass, construct line $BC \| OY$.

Step 5: Slide the straightedge into a position so that it

(i) meets BD at one marking,

(ii) meets BC at the other marking,

(iii) passes over the point O.

Step 6: Let Q be the point where the straightedge meets BC.

CLAIM: [OQ trisects ∠XOY.

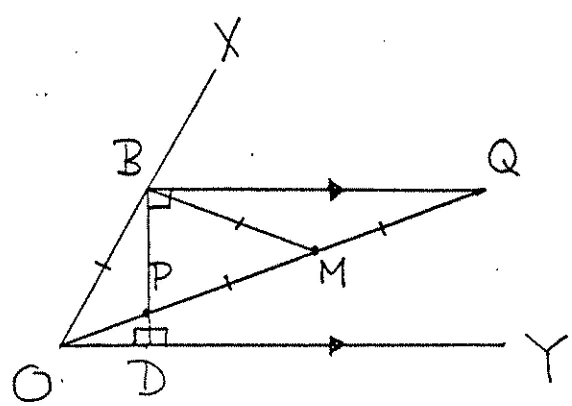
In order to prove this claim, we must establish the following result; this theorem was in fact known to Archimedes.

THEOREM 2. Let $d > 0$ be given, and let ∠XOY satisfy |∠XOY| < 90. Suppose that B ∈ (OX satisfies |OB| = d/2 and D ∈ (OY is the foot of the perpendicular from B to OY. Furthermore, suppose that P ∈ (BD), Q ∈ *INT* ∠XOY satisfy

(i) BQ‖OY,

(ii) |PQ| = d.

THEN we have |∠QOY| = |∠XOY|/3.



PROOF. Let M be the midpoint of $[PQ]$, so that

$$|PM| = |MQ| = |OB| = d/2.$$

CLAIM: $|MB| = d/2$ is also true. First notice ΔPBQ has a right angle at B; this is true because $BD \perp OY$ and $OY \| BQ$. But for right triangles ΔPBQ the midpoint M of the hypotenuse is equidistant from the vertices.

It follows that

$$(*) \quad |\angle BOM| = |\angle BMO| = 2|\angle MQB|$$

by standard theorems on isosceles triangles in Euclidean geometry (give the details!).

CLAIM: $P \in (OQ)$. First notice that $B\text{-}P\text{-}D$ is true by assumption, so that $P \in D$-side OB = Q-side OB. Hence $P \in Q$-side $OB \cap OQ = (OQ$. On the other hand, OD is parallel to BQ, so $O \in D$-side BQ. Also, $B\text{-}P\text{-}D$ implies $P \in D$-side BQ. Hence $P \in O$-side PQ, so that $P \in (QO$ also. It follows that $P \in (OQ \cap (QO = (OQ)$.

It then follows that $M \in (OQ)$ also, so that $\angle MQB = \angle OQB$. By the Crossbar Theorem, B and Y lie on opposite sides of $OP = OQ$, and hence $\angle MQB = \angle OQB$ and $\angle QOY$ are alternate interior angles. Hence

$$|\angle MQB| = |\angle QOY|$$

by the Alternate Interior Angle Theorem.

Therefore we have

$$|\angle XOY = \angle BOD| = |\angle BOM = \angle BOQ| + |\angle QOY|.$$

But by (*) the first summand in the latter expression equals $2|\angle QOY|$, so, it follows that

$$|\angle XOY| = 3|\angle QOY|, \quad \text{or} \quad |\angle QOY| = |\angle XOY|/3 \blacksquare$$

One point in this construction deserves further comment. although it is physically clear that we can slide the straightedge into the position described at Step 4, to be careful one should actually <u>prove</u> that this is possible. In other words one should prove that there exist points $P \in (BD)$, $Q \in BC \cap \text{INT} \angle XOY$ such that

(i) O,P,Q are collinear,

(ii) $|PQ| = d$.

This will be explained in the Footnotes to this section (the explanation will use some facts about continuous functions).

Cube Duplication

The key result may be stated as follows:

CONSTRUCTION 3. <u>Using marked straightedge and compass, it is possible to find</u> A,B,C <u>such that</u> $|AC|^3 = 2|AB|^3$.

This construction was known to Newton and Vieta.

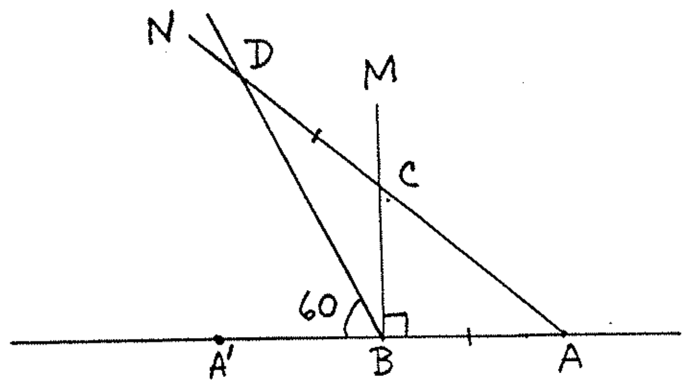
PROCEDURE. *Step 0:* Mark off $A \neq B$ with $|AB| = d$.

Step 1: Using (unmarked) straightedge and compass, locate M so that $|\angle ABM| = 90$.

Step 2: Locate A' such that A'-B-A and $|A'B| = |AB|$.

Step 3: Locate $N \in$ M-side AB so that $|\angle A'BN| = 60$ (and hence $|\angle ABN| = 120$).

Step 4: Slide the straightedge so that it passes over A, meeting (BM and (BN in C,D such that $|CD| = |AB| = d$.



CLAIM: $|AC|^3 = 2|AB|^3$.

(As with Construction 1, we shall justify the "sliding" of Step 4 in the Footnotes.)

The proof of the claim reduces to the following result:

THEOREM 4. <u>If</u> M,N <u>lie on the same side of line</u> AB <u>with</u>

$|\angle ABM| = 90$, $|\angle ABN| = 120$, <u>and</u> $|AB| = d$, <u>while</u>
$C \in$ (BM, $D \in$ (BN <u>satisfy</u> $|CD| = d$,

<u>THEN</u> $|AC|^3 = 2d^3$.

The first step is the following elementary fact:

LEMMA 5. <u>Given</u> ΔABC <u>with</u> $|\angle ABC| = 90$, $|\angle BAC| = 30$, $|\angle ACB| = 60$, <u>then</u> $|AB| = |BC|\sqrt{3}$.

PROOF OF LEMMA 5. Let $\Delta A'C'D'$ be an equilateral triangle with B' = midpoint $[C'D']$. Then $A'B' \perp C'D'$ (why?). Hence $\Delta ABC \sim \Delta A'B'C'$ by the A. A. Similarity Theorem. Hence it suffices to check $|A'B'| = |B'C'|\sqrt{3}$.

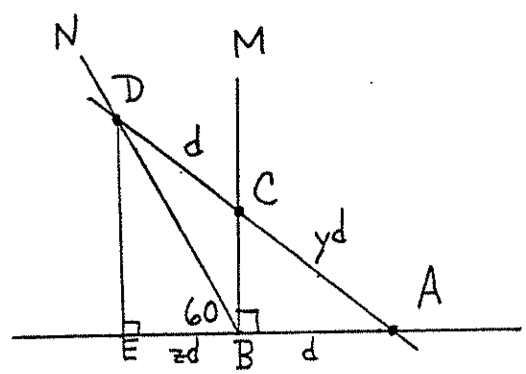
But $|B'C'| = |D'C'|/2 = |A'C'|/2$ (since $\Delta A'C'D'$ is equilateral). Hence

$$|A'B'| = \sqrt{|A'C'|^2 - |B'C'|^2 - |B'C'|^2} =$$

$$\sqrt{4|B'C'|^2 - |B'C'|^2} = \sqrt{3|B'C'|^2} =$$

$$|B'C'|\sqrt{3} \blacksquare$$

PROOF OF THEOREM 4. Let E be the foot of the perpendicular from D to AB. Then $|\angle DBA'| < 90$ implies $E \in (BA'$; hence $E\text{-}B\text{-}A$ is true.



Set $y = |AC|/d$, $z = |EB|/d$. Then $|ED| = zd\sqrt{3}$ by Lemma 5. The Basic Similarity Theorem then implies

$$\frac{d}{zd} = \frac{yd}{d}, \quad \text{so that} \quad y = \frac{1}{z}.$$

If we apply the Pythagorean Theorem to right triangle ΔAED, we find that

$$(d+yd)^2 = (d+zd)^2 + (\sqrt{3}zd)^2$$

or

$$d^2(1+2y+y^2) = d^2(1+2z+z^2+3z^2)$$

or

$$1 + 2y + y^2 = 1 + 2z + 4z^2.$$

If we substitute $y = 1/z$ in the last equation, multiply both sides by y^2, and simplify, we obtain the equation

$$2y^3 + y^4 = 2y + 4,$$

or

$$0 = y^4 + 2y^3 - 2y - 4 = (y^3-2)(y+2).$$

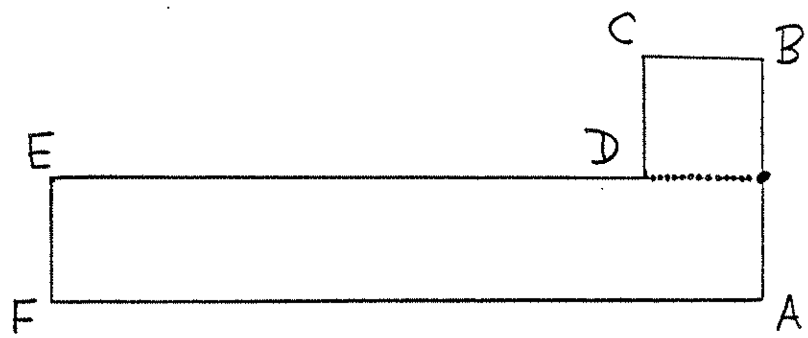
The only positive real root of this equation is $\sqrt[3]{2}$; since y is positive by definition, it follows that $y = \sqrt[3]{2}$ and hence that

$$\frac{|AC|}{|AB|} = \frac{yd}{d} = y = \sqrt[3]{2} \quad \blacksquare$$

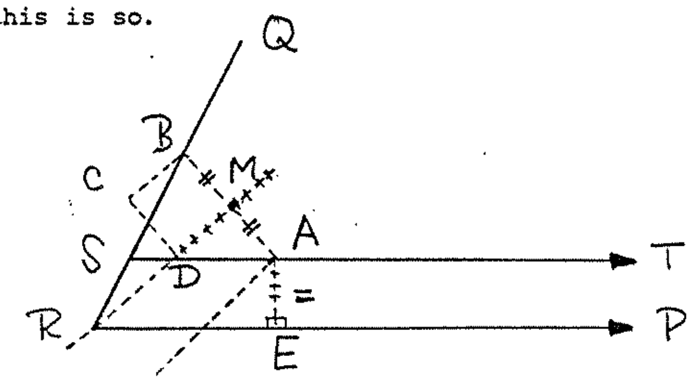
The exercises contain other methods for trisecting an angle and doubling a cube.

EXERCISES

1. A carpenter is able to trisect an angle by means of a special type of carpenter's square shown in the figure below. All the angles are right angles and $|EF| = |CD| = |AB|/2$.



To trisect ∠PRQ the carpenter first uses the longest edge to draw a ray [ST parallel to [RP at distance |EF|. Then placing the carpenter's square so that DE contains RA is on [ST, and B is on [RQ, the carpenter knows that [RD and [RA trisect ∠PRQ. Prove that this is so.

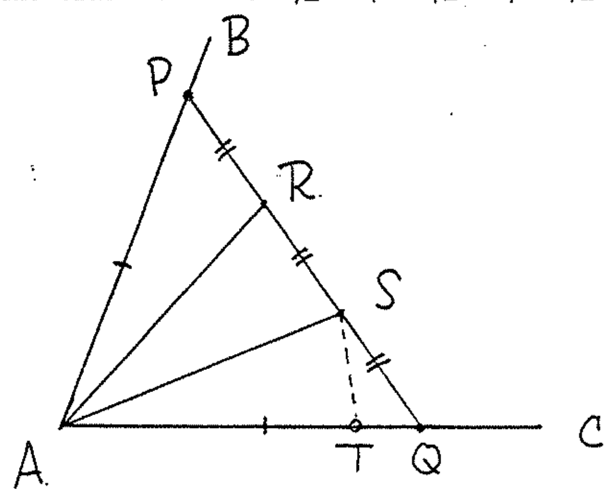


(*HINTS*: The ray [RD is the perpendicular bisector of [AB] and in fact contains the midpoint of [AB]; hence [RD bisects ∠ARB. Next, prove that A is equidistant from [RD and [RP; it then follows that [RA bisects ∠DRP. The key relations are |AB| = 2|EF| and |EF| = distance from A to PQ.)

2. Consider the parabolas defined in the Cartesian plane by the equations $x^2 = y$ and $y^2 = 2x$. Show that these parabolas intersect at the point $(2^{1/3}, 4^{1/3})$.

NOTE: This result says that if we enlarge our collection of allowable constructions to include parabolas, then it is possible to duplicate the cube using straightedge, compass, and a parabola plotter.

3. The following is a standard mistake in attempting to trisect an angle: Given $\angle BAC$, let P and Q be points on (AB and (AC such that $|AP| = |AQ|$. By straightedge and compass one can trisect [PQ], finding R,S so that P-R-S-Q and $|PR| = |RS| = |SQ| = |PQ|/3$. Then the claim is that $|\angle PAR| = |\angle RAS| = |\angle SAQ|$.



It is easy to show $|\angle PAR| = |\angle SAQ|$ (how?), but $|\angle RAS|$ is not equal to this common value. Fill in the reasons for the steps in the argument below.

(1) $|\angle APQ| = |\angle AQP|$.

(2) $\Delta APR \cong \Delta AQS$.

(3) $|AR| = |AS|$.

Now assume $|\angle SAQ| = |\angle RAS|$.

(4) $|\angle RSA| > |\angle AQS|$.

(5) $|\angle RSA| = |\angle SRA|$.

(6) $|\angle RSA| = 90 - (|\angle SAR|/2) < 90$.

(7) $|\angle QSA| > 90 > |\angle SQA|$.

(8) $|AQ| > |AS|$.

(9) There is a point $T \in (AQ)$ such that $|AT| = |AS|$.

(10) If $|\angle SAQ| = |\angle RAS|$, then $\Delta TAS \cong \Delta SAR$.

(11) $|ST| = |RS|$ and $|ST| = |SQ|$.

(12) $|\angle STQ| = |\angle SQT|$.

(13) $|\angle STQ| < 90$.

(14) $\angle SQT = \angle SQA$ (as sets).

(15) $|\angle STQ| < 90$.

(16) $|\angle STQ| + |\angle STA| < 180$.

(17) But Q-T-A is true.

(18) *Therefore* $|\angle STQ| + |\angle STA| = 180$.

(19) This is a contradiction.

(20) Therefore $|\angle SAQ|$ and $|\angle RAS|$ cannot be equal.

(Source: [Byrkit and Waters])

4. Prove that $z = 2\cos(180°/7)$ is a root of the cubic polynomial

$$x^3 + x^2 - 2x - 1 = 0.$$

(HINTS: Let $w = \exp(2\pi i/7)$. Then $w^7 = 1$ and also

$$0 = (w^7-1)/(w-1) = 1 + w + \ldots + w^6.$$

Use these and the Euler formula

$$2\cdot\cos(2\pi i/7) = \exp(2\pi i/7) + \exp(-(2\pi i/7)) = w + w^{-1}$$

to verify that z is a root of the polynomial in question. Notice that $w^{-1} = w^6$, $w^{-2} = w^5$, etc.)

Using this, explain why a regular 7-gon is not constructible by straightedge and compass. (Show that the cubic polynomial above has no rational roots, and apply the results in Moise.)

5. Let ΔABC be a triangle such that $AB \perp BC$, $|AC| = 3$, $|AB| = 2$ (hence $\cos|\angle BAC| = 2/3$). Prove that it is impossible to trisect $\angle BAC$ by straightedge and compass. You may use the formula $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$.

6. Let $A \neq B$ be such that $s = |AB|$ is rational. Prove that it is impossible to construct, by straightedge and compass, points D,E,F,G such that $|DE| = m$, $|FG| = n$, and

$$\frac{s}{m} = \frac{m}{n} = \frac{n}{2s} .$$

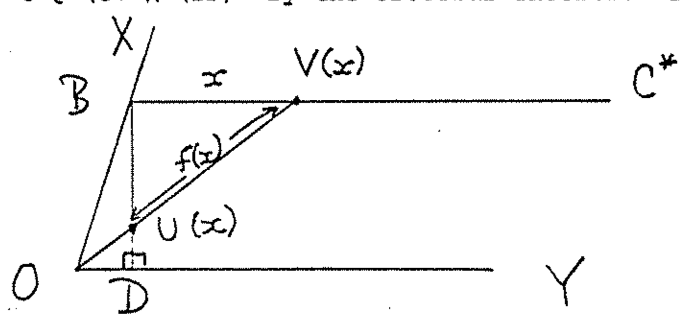
FOOTNOTES

Justification for Construction 1 and 3

We noted that the sliding ruler assertions (Step 5 in Construction 1 and Step 4 in Construction 2) required proof. Here are the details.

LEMMA 6. Given the setting of Construction 1, let B, B', C, D be defined as in steps 1-4. Then there exist points $P \in (BD)$ and $Q \in INT\ \angle XOY \cap BC$ such that $|PQ| = d$.

PROOF. *(Sketch)*. Let $C^* \in BC \cap$ Y-side OX. Let $x > 0$, and let $V \in (BC^*$ satisfy $|BV| = x$. Then $V \in INT\ \angle XOY$ and hence there is a point $U \in (OV \cap (BD)$ by the Crossbar Theorem. Set $f(x) = |UV|$.



CLAIM: $f(x)$ is a continuous function of x. Let $c = |BD|$, $y = |VD|$, $a = |OD|$. We then have

$$f(x) = \sqrt{(c-y)^2+x^2},$$

$$\frac{c-y}{x} = \frac{y}{a}$$

by the Pythagorean Theorem and the relation $\Delta UOD \sim \Delta UVB$. It follows that

$$y = \frac{ac}{x+a},$$

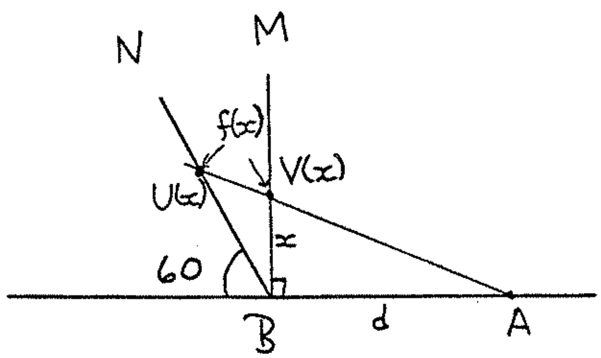
and hence $f(x)$ is clearly continuous in x. In fact, one also has that $f'(x) > 0$ for all $x > 0$ and

$$\lim_{x \to 0^+} f(x) = 0.$$

On the other hand, it is clear that $f(d) \geq d$ since $f(x) \geq x$ for all $x > 0$ (why?). If $f(d) = d$, then we have $|UV| = d$ if $|BV| = d$ and the result is proved. If $f(d) > d$, then by continuity we know that $d = f(z)$ for some z satisfying $0 < z < d$. Thus if we take V such that $|BV| = z$, we shall have $|UV| = d$ as required ■

LEMMA 7. Given the setting of Construction 3, let A,B,M,A',N be given as in Steps 0-3. Then there exist points $C \in (BM$ and $D \in (BN$ such that A,C,D are collinear and $|CD| = |AB| = d$.

PROOF. (*Sketch*) Let $x > 0$, and let $V \in (BM$ satisfy $|BV| = x$. If $x \leq d$, then $|\angle BVA| \geq |\angle BAV|$ by the Scalene Triangle Theorem, and hence (since $|\angle ABV| = 90$) we have $|\angle BAV| < 45$. Therefore $(AV$ and $(BN$ meet by Euclid's Fifth Postulate (for $|\angle ABN| + |\angle BAV| < 120 + 45 = 165$). Define U to be the point of intersection, and set $f(x) = |UV|$.



Let W be the foot of the perpendicular from U to AB. Since $|\angle UBA| = 120$, it follows that $U \in (BA'$. Set $y = |BW|$. By the Pythagorean Theorem, $UW \parallel VB$, and the Basic Similarity Theorem we have

(i) $$\frac{d}{\sqrt{d^2+y^2}} = \frac{y}{f(x)} .$$

On the other hand, since $|\angle UBW| = 60$ and $|\angle UWB| = 90$, by Lemma 5 we have

$$|UW| = \sqrt{3}\ |WB| = \sqrt{3}\ y.$$

Hence

(ii) $$\frac{y\sqrt{3}}{y+d} = \frac{x}{d} .$$

Combining (i) and (ii) we find that

$$f(x) = \frac{y\sqrt{d^2+y^2}}{d} ,$$

where $y = \dfrac{xd}{d\sqrt{3}-x}$.

Hence $f(x)$ is continuous once again. In fact,

$$\lim_{x \to 0} f(x) = 0$$

and

$$f(d) = a\,d\sqrt{1+a^2}, \quad \text{where,}$$

$$a = \frac{1}{\sqrt{3}-1}\,.$$

It follows that $f(d) > d$. Hence by continuity we must have $f(z) = d$ for some z satisfying $0 < z < d$ ■