An easy look at the cubic formula

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AN EASY LOOK AT THE CUBIC FORMULA

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Introduction

All students learn the quadratic formula for finding the roots of a quadratic equation. The cubic formula for solving cubic polynomials is seldom used, even though it has been known since the 1545 when Girolamo Cardano published his *Ars Magna* [2]. This cubic formula, like the quadratic formula, gives the exact answer in closed form. Fifty years ago, when this author was a schoolboy, algebra text books frequently included a detailed discussion of the cubic formula. Precalculus texts of today rarely consider the subject. Why? Because the cubic formula, unlike the quadratic formula, frequently involves awkward cube roots of complex numbers. Besides, excellent numerical methods are available, such as Newton's iterative method, which converge very rapidly to approximations with many accurate digits. However, there are cases where the exact closed form answer is appealing, and where the effort involved in using the cubic formula is not overwhelming.

It is the purpose of this brief note to show how the cubic formula can be presented easily at the precalculus level. While none of this material is new, the selection of items and their presentation is designed to avoid the difficulties mentioned above. We give a nice canonical form for the cubic formula that is relatively easy to remember. We show how to verify that the formula is correct, and we identify when it is profitable to use it.

The cubic formula in simplest form

To solve the cubic equation

(1) $$y^3 + py^2 + qy + r = 0$$

we must first remove the quadratic term. This is always achieved with the substitution

(2) $$y = x - \frac{p}{3}.$$

Substituting (2) into (1) we get

(3) $$x^3 - 3cx - 2a = 0,$$

where

(4) $$c = \frac{p^2}{9} - \frac{q}{3} \quad \text{and} \quad a = \frac{pq}{6} - \frac{p^3}{27} - \frac{r}{2}.$$

(Equation (3) is known as the *reduced cubic*.) Now we can write our cubic formula for the real root of (3).

Theorem:

Let

(5) $$b = a^2 - c^3 \geq 0.$$

Then a real root of $x^3 - 3cx - 2a = 0$ *is*

(6) $$x = \sqrt[3]{a + \sqrt{b}} + \sqrt[3]{a - \sqrt{b}}. \text{ (Cubic formula)}$$

We interpret all the roots in (6) as real numbers. (Actually, with proper interpretation of the radicals involved, formula (6) can give all three roots of (3) regardless of the values of the coefficients c and a. This does get messy when $b < 0$, and we will not consider that case here.)

Proof:

First notice that

(7) $$\sqrt[3]{a+\sqrt{b}}\,\sqrt[3]{a-\sqrt{b}} = \sqrt[3]{a^2-b} = \sqrt[3]{a^2-(a^2-c^3)} = c\,.$$

Cubing both sides of (6) we get

$$x^3 = \left(\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}}\right)^3$$

$$= \left(\sqrt[3]{a+\sqrt{b}}\right)^3 + 3\left(\sqrt[3]{a+\sqrt{b}}\right)^2 \sqrt[3]{a-\sqrt{b}} + 3\sqrt[3]{a+\sqrt{b}}\left(\sqrt[3]{a-\sqrt{b}}\right)^2 + \left(\sqrt[3]{a-\sqrt{b}}\right)^3$$

$$= a+\sqrt{b} + 3\sqrt[3]{a+\sqrt{b}}\,\sqrt[3]{a-\sqrt{b}}\left(\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}}\right) + a-\sqrt{b}$$

Using (6) and (7) we can rewrite this last expression as

$$x^3 = 3cx + 2a\,.$$

Thus we have verified that (6) is a root of (3) and the theorem is proved.

Examples

Example 1: Find a real root of $y^3 + 3y^2 + 6y + 2 = 0$.

Solution: Comparing our problem with (1), we see that $p = 3$ so we begin by making the

substitution (2) $y = x - p/3 = x-1$. This converts the original problem to $x^3 + 3x - 2 = 0$,

in which the quadratic term does not appear. Comparing this with (3) we see that $c = -1$

and $a = 1$. From (5) we get $b = a^2 - c^3 = 2$. Since $b > 0$ our theorem says that (6) gives us

a real root $x = \sqrt[3]{1+\sqrt{2}} + \sqrt[3]{1-\sqrt{2}}$. Since the second cube root is negative, it is best

written as $x = \sqrt[3]{1+\sqrt{2}} - \sqrt[3]{\sqrt{2}-1}$. Finally, a root of our original cubic is given by

$$y = x - 1 = \sqrt[3]{1+\sqrt{2}} - \sqrt[3]{\sqrt{2}-1} - 1\,.$$

Example 2: Find a real root of $y^3 - 7y^2 + 14y - 20 = 0$.

Solution: We compare our problem with (1) and see that $p = -7$. We start with the

substitution (2) $y = x - \dfrac{p}{3} = x + \dfrac{7}{3}$. Our original equation is now reduced to

$x^3 - 3cx - 2a = x^3 - \dfrac{7}{3}x - \dfrac{344}{27} = 0$ in which the quadratic term has been removed.

Comparing this last equation with (3) we see that $c = \dfrac{7}{9}$ and $a = \dfrac{172}{27}$. Calculating b from

(5) we get $b = a^2 - c^3 = \left(\dfrac{171}{27}\right)^2$. Since b is positive, a real root of our cubic is given by

(6) as $x = \sqrt[3]{\dfrac{172}{27} + \dfrac{171}{27}} + \sqrt[3]{\dfrac{172}{27} - \dfrac{171}{27}} = \dfrac{7}{3} + \dfrac{1}{3} = \dfrac{8}{3}$. Finally, a root of our original cubic is

given by $y = x + \dfrac{7}{3} = \dfrac{8}{3} + \dfrac{7}{3} = 5$. Since this root is an integer, it is easy to find the other

two roots by dividing our original cubic $y^3 - 7y^2 + 14y - 20$ by $y - 5$. This gives us the

quadratic $y^2 - 2y + 4 = 0$ which has roots $y = 1 \pm \sqrt{3}i$.

Example 3: Find a real root of $x^3 - x - 1 = 0$.

Solution: This problem comes from the interesting article [6] in which the "plastic

number" is defined as the root of our equation. Our equation has no quadratic term, so

there is no need to use the linear substitution. Comparing our equation with (3) we see

that $c = \dfrac{1}{3}$ and $a = \dfrac{1}{2}$. Using (4) we get $b = a^2 - c^3 = \dfrac{1}{4} - \dfrac{1}{27} = \dfrac{23}{108}$. Since b is positive,

we can use (6) to get the root

$$x = \sqrt[3]{\dfrac{1}{2} + \sqrt{\dfrac{23}{108}}} + \sqrt[3]{\dfrac{1}{2} - \sqrt{\dfrac{23}{108}}} = \dfrac{1}{6}\left(\sqrt[3]{108 + 12\sqrt{69}} + \sqrt[3]{108 - 12\sqrt{69}}\right).$$

Example 4: It is clear that $x = 1$ is a root of the cubic $x^3 + 3x - 4 = 0$. Use the cubic formula to obtain a surprising expression for this root.

Solution: Comparing our cubic with (3) we see at once that $c = -1$ and $a = 2$.

Calculating b we get $b = a^2 - c^3 = 5$. Our cubic formula now gives us

$x = \sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}} = \sqrt[3]{2 + \sqrt{5}} - \sqrt[3]{\sqrt{5} - 2}$. This does not look like $x = 1$, but a quick check with a calculator helps to convince us that it probably is 1. The reader might try to simplify this difference of two cube roots into the number 1, but all attempts to do this simply lead back to the original cubic $x^3 + 3x - 4 = 0$. The paper [4] discusses how to recognize when radical expressions of the form $\sqrt[n]{a + \sqrt{b}} + \sqrt[n]{a - \sqrt{b}}$, for $n = 2, 3, 4, \ldots$, reduce to simple numbers like integers or rational values.

Final remarks

There is much more that could be said about the cubic formula. How do you find the two complex roots when $b > 0$, and how do we find any roots when $b < 0$? To answer these questions requires a quantum leap in the difficulty of our presentation. This is not our purpose. We hope that we have shown that there is partial information about the cubic formula that is both interesting and useful. The reader can find complete presentations of this subject in many algebra text books dating from before 1960 such as [5] and nice summaries in handbooks such as [3].

References

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