11.B. A FEW EXAMPLES

LOGARITHMS. For more than three centuries, logarithms were an important computational tool. Although they have been generally superseded by electronic devices, there are still some contexts in which they are useful.

EXAMPLE. Determine the number of digits in $2^{300} \cdot 3^{200}$.

SOLUTION. We have

$$\left.\begin{aligned} \log_{10} 2 &= 0.301029996 \\ \log_{10} 3 &= 0.477121255 \end{aligned}\right\} \text{ so that}$$

$$\log_{10} 2^{300} \cdot 3^{200} = 300 \log_{10} 2 + 200 \log_{10} 3 = 185.7332496$$

Therefore there are 186 digits in $2^{300} \cdot 3^{200}$. ∎

LOCUS PROBLEMS AND COORDINATES.

EXAMPLE 1. Given a line L in the plane find the set of all points p such that distance $(p, L) = a > 0$.

SOLUTION. Choose coordinates so that L corresponds to the x-axis, and let $p = (x, y)$. Then the distance from p to L is $|y|$. Hence the locus set consists of all $p = (x, y)$ such that $|y| = a$, or equivalently $y = \pm a$. Hence the set is the points on the two parallel lines $y = \pm a$.

See locus-problems.pdf and locus-problems1.pdf for comments on comparing proofs of this sort via coordinates to classical proofs via synthetic Euclidean geometry.

<u>EXAMPLE 2.</u> Let L and M be parallel lines. Then the set of all points p such that distance (p, L) = 4 distance (p, M) is a pair of lines parallel to L and M.

<u>SOLUTION</u> Choose coordinates so that L corresponds to the x-axis (equation $y = 0$) and one ($\Rightarrow$ all) ~~con~~ y-coordinate(s) for point(s) of M are positive. Hence M has defining equation $y = d$. Then the point set is defined by

$$\underbrace{|y|}_{\text{distance }(p, L)} = \underbrace{4|y-d|}_{4 \cdot \text{distance }(p, M)}$$

$$\cancel{y^2 = 16y^2 - 32yd + 16d^2}$$

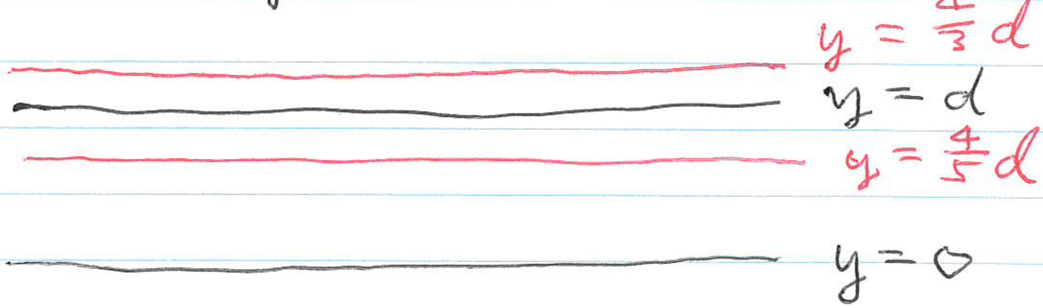
$$\cancel{0 = 15y^2 - 32yd + 16d^2}$$

$$4(y-d) = \pm y \quad \underline{\text{Two cases}}$$

$\underline{+}$ $4(y-d) = y \Rightarrow 3y - 4d = 0 \Rightarrow$

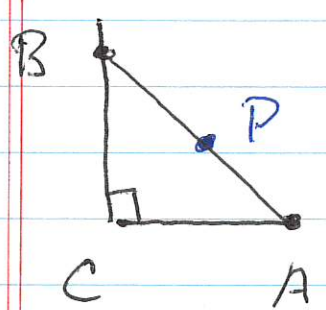
$y = \frac{4}{3}d$. (one line)

$$= \quad 4(y-d) = -y \Rightarrow 5y - 4d = 0$$
$$\Rightarrow y = \tfrac{4}{5}d. \text{ (other line)}$$



The algebra shows there are TWO lines.

<u>EXAMPLE 3.</u> (Sloping ladder problem).



Find the set of all P so that P is a midpoint of [AB] if the length d of [AB] is constant.

<u>SOLUTION.</u> Choose coordinates so that

$$C \leftrightarrow (0,0), \quad \underset{x>0}{A \leftrightarrow (x,0)}, \quad \underset{y>0}{B \leftrightarrow (0,y)}$$

We then have $x^2 + y^2 = d^2$

and $P = \left(\frac{x}{2}, \frac{y}{2}\right)$.

Write $P = (u, v)$ so $x = 2u$, $y = 2v$.

Then $x^2 + y^2 = d^2 \Longrightarrow u = \sqrt{\left(\frac{d}{2}\right)^2 - v^2}$

By construction x & y ($\Rightarrow u$ & v) are ≥ 0,

Hence the locus is the 90° arc of the circle $u^2 + v^2 = \left(\frac{d}{2}\right)^2$ whose endpoints are $\left(\frac{d}{2}, 0\right)$ and $\left(0, \frac{d}{2}\right)$.