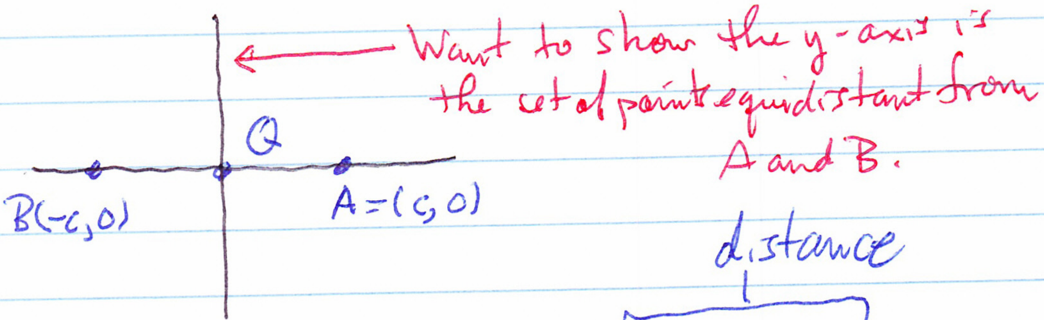
MORE LOCUS PROBLEMS

Here is a proof of a standard example:

THEOREM. Let $A \neq B$ be distinct points in the plane. Then the locus (= set) of all points in the plane which are equidistant from A and B is the perpendicular bisector of the closed segment [AB].

PROOF. Choose c so that $2c = \text{distance}(A,B)$, and let Q be the midpoint of [AB]. Now choose a coordinate system so that the line of A, B, Q is the x-axis and Q is the origin. Then the $\perp$ bisector of [AB] is the y-axis and the coordinates of A, B are $(c,0)$ and $(-c,0)$.



CLAIM If $P=(x,y)$, then $\underbrace{d(A,P)}_{\text{distance}} = d(B,P) \iff x=0$. This will prove the theorem since the y-axis is defined by the eqn. $x=0$.

The distances from P to A and B are given by

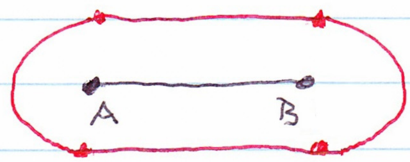
$$d(A,P) = \sqrt{(x-c)^2 + y^2}$$
$$d(B,P) = \sqrt{(x+c)^2 + y^2}$$

Therefore $d(A,P) = d(B,P) \Longleftrightarrow d(A,P)^2 = d(B,P)^2 \Longleftrightarrow (x-c)^2 + y^2 = (x+c)^2 + y^2$. Subtracting like terms from each side of the latter equation (after using $(u+v)^2 = u^2 + 2uv + v^2$), we see that this equation reduces to $-2cx = 2cx$. Since $c \neq 0$, the equation $-2cx = 2cx$ is equivalent to $-x = x$, which in turn is equivalent to $x = 0$. Therefore $d(A,P) = d(B,P) \Longleftrightarrow x = 0$; i.e., if and only if P lies on the $\perp$ bisector. ∎

As the example on the next page illustrates, it is not difficult to formulate locus problems whose answers are relatively complex:

PROBLEM. Suppose we are given a closed segment [AB] and a positive real number d. What is the locus (= set) of all points X such that the (least) distance from X to a point of [AB] is equal to d?

APPROACH. Start by drawing a picture:



Educated guess for describing the locus.

This drawing suggests that there are 4 pieces to the locus: Two semicircular arcs whose centers are A + B respectively, and two closed segments on lines $\parallel$ AB. We need to verify this, preferably using coordinate geometry.

Choose coordinates as in the preceding result, so that $A \leftrightarrow (-c, 0)$, $B \leftrightarrow (c, 0)$. Then the proof splits into 3 cases depending upon whether a point $P = (x, y)$ satisfies $x \leq -c$, $-c \leq x \leq c$, or $x \geq c$. (Details are left to the reader!)