Solutions to two locus problems

Here are the solutions to two problems posed in other documents:

PROBLEM 1. *Let $A \neq B$ be the points in the coordinate plane with coordinates $(-c, 0)$ and $(c, 0)$ where $c > 0$, let $[AB]$ denote the closed segment joining A to B, and let $d > 0$. Then the locus (= **set**) of all points X such that the (minimum) distance from X to $[AB]$ equals D consists of the following:*

(a) *The semicircular arc centered at B which passes through $(c + d, 0)$ (hence its radius is d and its endpoints are $(c, \pm d)$).*

(b) *The semicircular arc centered at A which passes through $(-c - d, 0)$ (hence its radius is d and its endpoints are $(-c, \pm d)$).*

(c) *The closed line segment joining $(-c, d)$ to (c, d).*

(d) *The closed line segment joining $(-c, -d)$ to $(c, -d)$.*

There is a drawing of this locus on the last page of `locus-problems2.pdf`.

PROBLEM 2. *Let $A \neq B$ be the points in the coordinate plane with coordinates $(0, 0)$ and $(a, 0)$, where $a > 0$. Then the locus (= **set**) of all points P such that the distance from A to P is* **twice** *the distance from B to P is the circle with center $\left(0, \frac{4}{3}\, a\right)$ and radius $\frac{2}{3}\, a$.*

There is a drawing of this locus on the last page of this document.

Notation. Given two points U and V, we shall denote the distance between U and V by $\mathbf{d}(U, V)$.

Solution to the first problem

It is necessary to split the solution into three cases, depending upon which of the following holds:

(1) $P = (x, y)$, where $x \leq -c$.

(2) $P = (x, y)$, where $-c \leq x \leq c$.

(3) $P = (x, y)$, where $c \leq x$.

In the first case, we need to prove that P lies on the locus if and only if P lies on the semicircular arc centered at A. In the second case, we need to prove that P lies on the locus if and only if P lies on one of the two line segments. In the third case, we need to prove that P lies on the locus if and only if P lies on the semicircular arc centered at B.

The preceding three statements will be immediate consequences of the following assertions:

If $P = (x, y)$ with $x \leq -c$, and $Y = (0, t)$ lies on the interval $[AB]$, then $\mathbf{d}(P, Y) \geq \mathbf{d}(P, A)$ with equality if and only if $Y = A$.

If $P = (x, y)$ with $-c \leq x < c$, and $Y = (0, t)$ lies on the interval $[AB]$, then $\mathbf{d}(P, Y) \geq \mathbf{d}(P, Z)$, where $Z \in [AB]$ is the point $(x, 0)$, with equality if and only if $Y = Z$.

If $P = (x, y)$ with $c \leq x$, and $Y = (0, t)$ lies on the interval $[AB]$, then $\mathbf{d}(P, Y) \geq \mathbf{d}(P, B)$ with equality if and only if $Y = B$.

We shall now explain why the second list of statements implies the first list. Let d be the shortest distance from P to a point of $[AB]$.

In the first case, the shortest distance $\mathbf{d}(P, Y)$ must occur when $Y = A$, and therefore if $x \leq -c$ we have $d = \mathbf{d}(P, A)$. This means that the point P must lie on the semicircular arc defined by $x \leq -c$ and $(x + c)^2 + y^2 = d^2$. In the second case, the shortest distance $d = \mathbf{d}(P, Y)$ must occur when $Y = Z$, and therefore if $x \leq -c$ we have $d = \mathbf{d}(P, A)$. This means that the point P must lie on one of the two closed segments defined by $-c \leq x < c$ and $|y| = d$; of course, the latter is equivalent to $y = \pm d$. In the third case, the shortest distance $\mathbf{d}(P, Y)$ must occur when $Y = B$, and therefore if $c \leq x$ we have $d = \mathbf{d}(P, B)$. This means that the point P must lie on the semicircular arc defined by $c \leq x$ and $(x - c)^2 + y^2 = d^2$.

Proofs of the assertions about distances. Suppose that $x \leq -c$. Since Y lies on the segment $[AB]$ we must have $-c \leq t$. Subtracting x from these inequalities yields $0 \leq -c - x \leq t - x$, which immediately yields the inequality

$$\mathbf{d}(P, Y) \quad = \quad \sqrt{(t - x)^2 + y^2} \quad \geq \quad \sqrt{(-c - x)^2 + y^2} \quad = \quad \mathbf{d}(P, A) \ .$$

Furthermore, equality holds if and only if $t = -c$, or equivalently $Y = A$.

Next, suppose that $-c \leq x < c$. Then we have the inequality

$$\mathbf{d}(P, Y) \quad = \quad \sqrt{(t - x)^2 + y^2} \quad \geq \quad |y| \quad = \quad \mathbf{d}(P, Z)$$

with equality if and only if $t = x$; *i.e.*, equality holds if and only if $Y = (x, 0)$.

Finally, suppose that $c \leq x$. Since Y lies on the segment $[AB]$ we must have $t \leq c$. Subtracting t from these inequalities yields $0 \leq c - t \leq x - t$, which immediately yields the inequality

$$\mathbf{d}(P, Y) \quad = \quad \sqrt{(x - t)^2 + y^2} \quad \geq \quad \sqrt{(c - x)^2 + y^2} \quad = \quad \mathbf{d}(P, B) \ .$$

Furthermore, equality holds if and only if $t = c$, or equivalently $Y = B$.

As indicated earlier in the argument, this completes the proof.■

Solution to the second problem

We are given $A = (0, 0)$ and $B = (a, 0)$ with $a > 0$. The locus is then defined by the equation $\mathbf{d}(P, A) = 2 \cdot \mathbf{d}(P, B)$. Since both of the distances are nonnegative, the latter is equivalent to the squared equation $\mathbf{d}(P, A)^2 = 4 \cdot \mathbf{d}(P, B)^2$, which in turn reduces to the following equation involving coordinates:

$$x^2 \ + \ y^2 \quad = \quad 4 \cdot \left((x - a)^2 \ + \ y^2 \right)$$

If we subtract $x^2 + y^2$ from both sides and then divide both sides by 3, we obtain the following equivalent equation for the locus:

$$0 \quad = \quad x^2 \ - \ \frac{8ax}{3} \ + \ \frac{4a^2}{3} \ + \ y^2$$

In order to express this as the equation of a circle, we need to complete the square in the first three terms, replacing them by $(x-b)^2 - c^2$ for some real number b and some $c^2 > 0$. If we complete the square, we find that the right hand side of the preceding display is given by the following:

$$0 \;=\; x^2 - \frac{8ax}{3} + \frac{16a^2}{9} + \frac{4a^2}{3} - \frac{16a^2}{9} + y^2 \;=\; \left(x - \frac{4a}{3}\right)^2 - \left(\frac{2a}{3}\right)^2 + y^2$$

Since the right hand side is the equation of a circle with center $\left(0, \frac{4}{3}\,a\right)$ and radius $\frac{2}{3}\,a$, this completes the proof.■

Generalization of the second problem

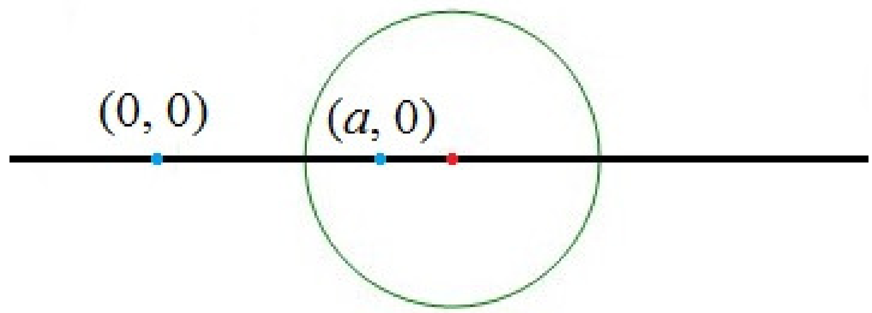
One natural generalization of this problem is to consider the locus of all points P such that $\mathbf{d}(A,P) = \rho \cdot \mathbf{d}(B,P)$ for some constant $\rho > 1$. An argument like the preceding one shows that the locus in this case is again a circle. If we choose coordinates as in the solution to the second problem, the center of this circle is the line joining A to B, and the circle meets this line at the points

$$\left(\frac{a\rho}{\rho+1},\, 0\right) \qquad \text{and} \qquad (a\rho, 0)\ .$$

Of course, the center of the circle will be the point halfway between the latter, and the radius is half the difference between the first coordinates of the two points. Computations of the explicit coordinates for the center point and the value of the radius are routine exercises in algebra.

Note also that the argument does **not** generalize to the case $\rho = 1$, for if the latter holds then the equation $x^2 + y^2 = (x-a)^2 + y^2$ does not yield a quadratic polynomial in x and y if we subtract $x^2 + y^2$ from both sides.■

Drawing for the second problem



The two points are indicated by blue dots, the locus is the circle in the drawing, and the center of the circle is indicated by the red dot. As noted above, the center of the circle has coordinates $(4a/3, 0)$ and the radius of the circle is equal to $2a/3$. This circle meets the $x-$ axis at the points $(2a/3, 0)$ and $(2a, 0)$.