Addendum to "Classical transcendental curves"

Here are some illustrations to accompany the online document named above, which is available at the following address:

http://math.ucr.edu/~res/math153/transcurves.pdf

For the sake of completeness, here is another related reference:

http://math.ucr.edu/~res/math144/transcendentals.pdf

At one point in the discussion of the ***Archimedean spiral***, we claim that every vertical or horizontal line meets the spiral at infinitely many points and mention that this seems clear if one plots the graph of the spiral and draws vertical or horizontal lines. The following picture suggests this observation:

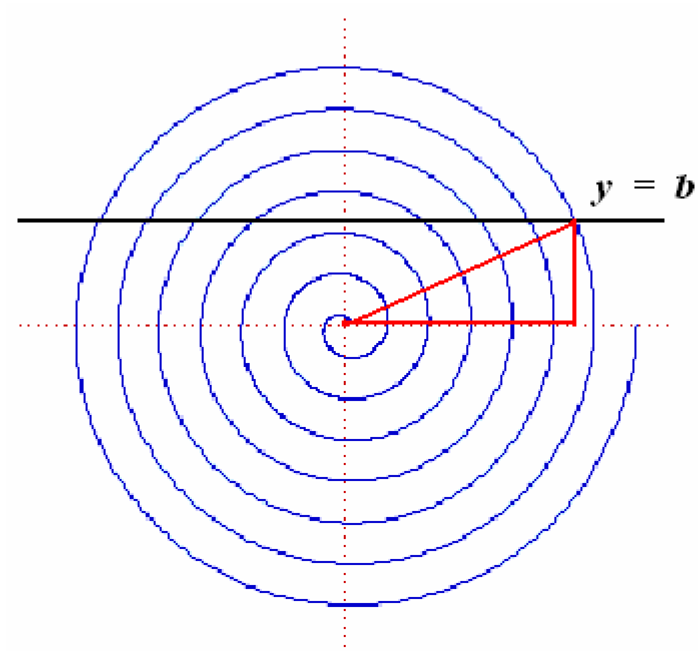

Figure 1

In order to prove this assertion for horizontal lines, it is necessary to show that systems of equations like $r = \theta$ and $y = b$ always have infinitely many distinct solutions, and for vertical lines one has a similar system with x replacing y. If $b = 0$ the existence of infinitely many solutions can be checked directly, and in the other cases one needs to analyze the situation more carefully. The next figure gives more detailed information about a portion of the previous one.

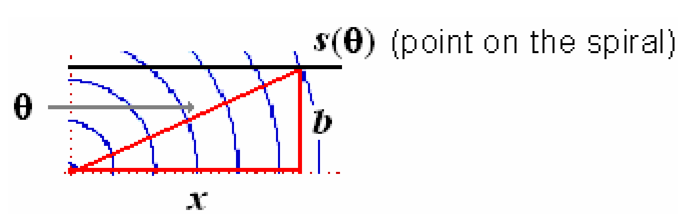

Figure 2

Here is an explanation of Figure **2:** Notice that we have a right triangle and we know that the length of the vertical side is the absolute value of b; we need to find the length of the horizontal side, and to do so we need to find the length of the hypotenuse. If we take the standard parametrization of the spiral and choose the parameter θ so that the point we want is $s(\theta)$, then it follows that the length of the hypotenuse – which is also the distance between the origin and the point on the curve – will be equal to θ, so that the latter must satisfy $\theta^2 = x^2 + b^2$. Furthermore, the defining equation for the spiral now tells us that the measure of the vertex angle at the origin is essentially given by θ; this is true precisely if the point lies in the first quadrant of the coordinate plane, and elsewhere the angle is $\pi - \theta$ in the second quadrant, $\theta - \pi$ in the third and $-\theta$ in the fourth. Regardless of the quadrant in which the curve point lies, this means that the **cotangent** of θ is equal to the quotient x/b, and hence it leads to the equation for x

$$x = b \cdot \cot \sqrt{x^2 + b^2}$$

which is stated in http://math.ucr.edu/~res/math153/transcurves.pdf.

The third and final figure illustrates the piece of the ***Quadratrix/Trisectrix of Hippias*** which was studied in classical Greek mathematics. As indicated in the document mentioned above, this is basically the graph of the function $x \cot x$ between $\pm\frac{1}{2}\pi$ with the point $(0, 1)$ added to obtain a function which is continuous (and in fact has a convergent Maclaurin series expansion) for all values of x between $\pm\frac{1}{2}\pi$.

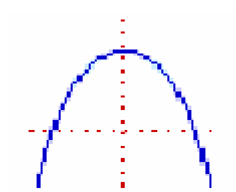

Figure 3

Note: Using results from the complex variable theory and the extended definitions for trigonometric functions for arbitrary complex numbers, one can in fact prove that the function $x \cot x$ has a convergent Maclaurin series expansion for all values of x between $\pm\pi$.