12.A. The proof of Cavalieri's Principle

As indicated in `history12.pdf`, Cavalieri's Principle is a powerful method for comparing the volumes of two solids in 3-space. The purpose of this document is to discuss the steps needed to prove this fact from the viewpoint of coordinate geometry and integral calculus. Although it is possible to formulate and prove a fairly general version of Cavalieri's Principle within the framework of an undergraduate real variables course, setting everything up properly is a fairly delicate thing to do, so we shall use the theory of **Lebesgue integration**, which is covered in basic graduate level courses on real variables and has been the standard approach to working with integrals in mathematics for the past hundred years. Many graduate level textbooks on real analysis or measure theory describe everything that we need, and the following book is one specific recommendation:

> R. Wheeden and A. Zygmund, **Measure and Integral — An Introduction to Real Analysis**. Marcel Dekker, New York, 1977.

The formulation and proof of Cavalieri's Principle involve a concept known as *measurability* for subsets of n-dimensional coordinate space $\mathbb{R}^n$. For our purposes it is enough to know that most subsets one encounters in standard geometrical problems or constructs by standard methods will be measurable. In more precise mathematical terms, if a problem asks for the area or volume of a subset in $\mathbb{R}^2$ or $\mathbb{R}^3$ this subset is almost certain to be a finite union of subsets defined by some collection of equations $f(x) = a$ and inequalities $g(x) \leq b$, where f and g are continuous, which means that it is a *closed* subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ (this means it is "closed under taking limits of convergent sequences"). Basic results on measurable sets imply that every such closed subset is measurable.

Cavalieri's Principle turns out to be a simple corollary of a basic integral formula for the volume of a 3-dimensional solid S in terms of the areas of the planar sections S_t formed by intersecting S with the horizontal planes $z = t$, where t runs through all real numbers which are third coordinates of points in S.

THEOREM. *Let $a < b$ be real numbers, and let S be a bounded measurable subset of $\mathbb{R}^3$ which lies between the parallel planes $z = a$ and $z = b$. Assume further that for each $t \in [a, b]$ the plane section set S_t corresponds to a measurable subset of $\mathbb{R}^2$ under the vertical projection sending (x, y, t) to (x, y). Then we have*

$$\text{Volume}(S) \quad = \quad \int_a^b \text{Area}(S_t)\, dt \ .$$

Before discussing the proof of this result, we shall indicate how Cavalieri's Principle follows from it.

COROLLARY. (Cavalieri's Principle) *Let $a < b$ be real numbers, let A and B be bounded measurable subsets of $\mathbb{R}^3$ which lie between the parallel planes $z = a$ and $z = b$, and suppose that A_t and B_t are the (measurable) plane sections as in the statement of the theorem. If* $\text{Area}(A_t) = \text{Area}(B_t)$ *for all $t \in [a, b]$, then* $\text{Volume}(A) = \text{Volume(B)}$.

Derivation of Cavalieri's Principle. Two applications of the theorem yield the equations

$$\text{Volume}(A) \quad = \quad \int_a^b \text{Area}(A_t)\, dt \ , \qquad \text{Volume}(B) \quad = \quad \int_a^b \text{Area}(B_t)\, dt \ .$$

We are given that $\text{Area}(A_t) = \text{Area}(B_t)$ for all $t \in [a, b]$, so it follows that the integrals on the right hand sides of these equations are equal, and hence the left hand sides must also be equal. Since the latter are the volumes of A and B, it follows that these volumes are equal.■

Proof of the theorem. Our argument is based upon the material in Chapter 6 of Wheeden and Zygmund. Let $J = [c_1, d_1] \times [c_2, d_2]$ be a solid rectangular region in $\mathbb{R}^2$ such that if $\mathbf{p} = (p_1, p_2, p_3)$ is in S then $p_i \in [c_i, d_i]$ for $i = 1, 2$. It follows that S is contained in the solid rectangular box $J^* = J \times [a, b] \subset \mathbb{R}^3$. Define the *characteristic function* χ_S of S on this rectangular box such that $\chi_S(\mathbf{p}) = 1$ if $\mathbf{p} \in S$ and $\chi_S(\mathbf{p}) = 0$ if $\mathbf{p} \notin S$. The measurability of S implies that S is a measurable function on the rectangular box J^* (this function is not continuous if S is a nonempty proper subset of J^*, but the function is sufficiently well behaved so that it can be integrated). Our assumptions imply the hypotheses Theorem 6.1 (*Fubini's Theorem*) on pages 77–78 of Wheeden and Zygmund, and in fact we are assuming a strong form of the first conclusion in Fubini's Theorem (the first conclusion only states that "almost all" of the sets S_t are measurable — the term "almost all" means there may be some exceptions that can be safely ignored). Therefore Fubini's Theorem implies that

$$\iiint \chi_S(x, y, t)\, dx\, dy\, dt \quad = \quad \int_a^b \left(\iint_J \chi_S(x, y, t)\, dx\, dy \right) dt \ .$$

Fundamental properties of integrals imply that the left hand side is the volume (or measure) of S, and likewise for each choice of t the inner integral on the right hand side is equal to the area (or measure) of S_t. If we substitute for the appropriate integrals using these equations, we obtain the desired identity

$$\text{Volume}(S) \quad = \quad \int_a^b \text{Area}(S_t)\, dt \ .■$$

FINAL REMARKS. Fubini's Theorem is named after G. Fubini (1879–1943). Additional background on Cavalieri's Principle can be found on pages 3–5 of `history12.pdf` and pages 156–159 (document pages 15–17) of the online reference cited there:

`http://math.ucr.edu/~res/math133/geometrynotes3c.pdf`