12.B. The binomial series

The goal is to derive the Newton binomial series for the function $(1+x)^a$, which is valid for every nonzero real value of a and all x such that $|x| < 1$.

$$(1+x)^a \quad = \quad \sum_{k=0}^{\infty} \binom{a}{k} x^k \quad \text{where} \quad \binom{a}{k} \quad = \quad \frac{a(a-1)\cdots(a-k+1)}{k!}$$

Derivation. Denote the power series on the right hand side by $B_a(x)$. Then the ratio test for convergence of infinite series implies that $B_a(x)$ converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. Exactly as in the case where a is a positive integer we have the identity

$$\binom{a}{k} \quad = \quad \binom{a-1}{k} + \binom{a-1}{k-1}$$

and this leads to the formula $B_a(x) = (1+x)B_{a-1}(x)$. Likewise, we have the identity

$$\binom{a}{k} \quad = \quad \frac{a}{k} \cdot \binom{a-1}{k-1} \qquad \text{or equivalently} \qquad k \cdot \binom{a}{k} \quad = \quad a \cdot \binom{a-1}{k-1}$$

which implies the differentiation formula $B'_a = aB_{a-1}$.

Now consider the function $Q(x) = (1+x)^{-a}\, B_a(x)$ and compute its derivative. By the standard differentiation rules and the preceding identities $Q'(x)$ is equal to

$$(1+x)^{-a} \cdot aB_{a-1}(x) \ + \ (-a)(1+x)^{-a-1} \cdot B_a(x) \quad =$$

$$(1+x)^{-a} \cdot aB_{a-1}(x) \ + \ (-a)(1+x)^{-a-1} \cdot (1+x)B_{a-1}(x)$$

and one can check directly that the right hand side equals zero. Therefore $Q(x)$ is constant, and since $Q(0) = 1$ we see that $1 = (1+x)^{-a}\, B_a(x)$. If we multiply both sides of this equation by $(1+x)^a$, we obtain Newton's binomial formula.■