14.D. An unusual smooth function

In the preceding file **(14.C)** we mentioned that the antiderivative of $\exp(-ax^2)$ cannot be expressed finitely in terms of the standard functions in elementary calculus, but it does have a fairly simple and extremely useful infinite series expansion. Our purpose here is to discuss a function with diametrically opposite properties: It can be expressed in fairly simple terms and it is infinitely differentiable everywhere, but it cannot be expressed as a convergent power series near $x = 0$. Most of the material below is taken from the following online site:

http://planetmath.org/encyclopedia/InfinitelyDifferentiableFunctionThatIsNotAnalytic.html

If f is an infinitely differentiable function at $x = a$, then we can certainly write a Taylor of Maclaurin series for f at $x = a$ using the higher order derivatives $f^{(n)}(a)$, where $n \geq 0$. However, it does **not** necessarily follow that the power series for f actually converges to f, as the following example shows:

Define f by the conditions

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Then f is an infinitely differentiable function, and for all nonnegative integers n we have $f^{(n)}(0) = 0$ (see below). Therefore the Maclaurin series for f at $x = 0$ is just 0. Since $f(x) > 0$ when x is nonzero, clearly this series does not converge to f.

Proof that $f^{(n)}(0) = 0$

Let $p(x)$ and $q(x)$ be polynomials with real coefficients, let f as above, and define

$$g(x) = \frac{p(x)}{q(x)} f(x).$$

Then for all nonzero values of x we have

$$g'(x) = \frac{(p'(x) + 2x^{-3}p(x))q(x) - q'(x)\,p(x)}{q^2(x)} \exp(-1/x^2)$$

If we now apply L'Hospital's Rule and the Mean Value Theorem, we see that

$$g'(0) = \lim_{x \to 0} g'(x) = 0.$$

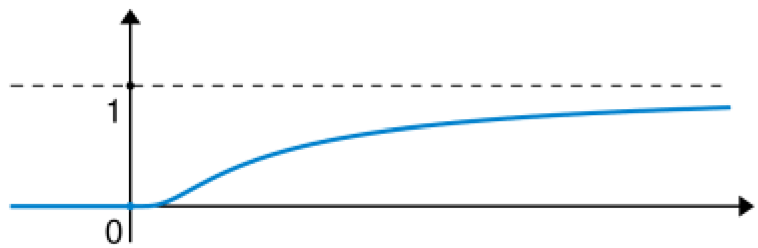
If we set $p_0(x) = q_0(x) = 1$, then the preceding discussion recursively yields sequences of polynomials $p_n(x)$ and $q_n(x)$ such that for all nonzero values of x we have

$$f^{(n)}(x) = \left(\frac{p(x)}{q(x)}\right) f(x).$$

Furthermore, it follows that $f^{(n)}(0) = 0$, which is what we wanted to show.

Useful properties of this function

The unusual behavior of the function f at $x = 0$ turns out to be important for many purposes, for because it yields some infinitely differentiable functions which are not constant functions but are ***constant on bounded or unbounded closed intervals***. For example, consider the function $g(x)$ which is equal to $f(x)$ when $x \geq 0$ and is set equal to zero for $x < 0$. The graph of this function is sketched in the drawing below; note that the function is positive when x is positive and as x approaches 0 from the positive side it corresponds to an extremely soft landing of an airplane.



(Source: http://upload.wikimedia.org/wikipedia/commons/b/b4/Non-analytic_smooth_function.png)

It follows that g is constant for nonpositive values of x, but g is infinitely differentiable for all values of x, and $g^{(n)}(0) = 0$ for all $n \geq 0$. Further examples are described in the exercises for this unit.