REWRITING SOLUTIONS TO LOGISTIC EQUATIONS

In the lectures the logistic differential equation for population growth was stated in the form

$$y' \quad = \quad a\,y \; - \; b\,y^2$$

and the general solution was written in the form

$$y \quad = \quad \frac{Ke^{ax}}{1 + cKe^{ax}}$$

where $c = b/a$ and K is a constant of integration. Note that the constant of integration and the initial condition are related by the equation

$$y(0) \quad = \quad \frac{K}{1 + cK}$$

or equivalently by the solution of the latter for K:

$$K \quad = \quad \frac{y(0)}{1 - cy(0)}$$

In Exercise 9 on page 518 of the text, the equation is written in the form

$$y' \quad = \quad r(M - y)y$$

where $r = 1.5 \times 10^{-3}$ and $M = 150$, and we claim that the solution with initial condition $y(0) = 6$ can be rewritten in the form

$$y \quad = \quad \frac{150}{1 + 24e^{-0.225\,x}} \;.$$

To check this, we must express M, r and K in terms of the numbers given above. First of all, we have that $b = r$ and $a = rM$, from which $c = 1/M$ follows immediately. Substituting our values for r and M as above, we see that $c = 1/150$ and $a = 0.225$. Now we may use the value for c together with $y(0) = 6$ to see that

$$K \quad = \quad \frac{y(0)}{1 - cy(0)} \quad = \quad \frac{My(0)}{M(1 - cy(0)\,)} \quad = \quad \frac{My(0)}{M - y(0)} \quad = \quad \frac{900}{144} \quad = \quad \frac{150}{24} \;.$$

With all this information, we can now rewrite the solution as

$$y \quad = \quad \frac{Ke^{ax}}{1 + cKe^{ax}} \quad = \quad \frac{24Ke^{ax}}{24 + 24cKe^{ax}} \quad = \quad \frac{150e^{ax}}{24 + e^{ax}} \quad = \quad \frac{150}{24e^{-ax} + 1}$$

(since $24K = 150$ and $24cK = 1$), and if we substitute $a = 0.225$ we obtain the form of the solution given above.■

IMPORTANT NOTE. The form of the solution in **Exercise 14, page 519**, is often the most convenient when the equation is written as $y' = r(M - P)P$.