The maximality property of the real number system

The purpose of this document is to explain how one can establish the folowing characterization of real numbers. We shall follow the notation and definitions in `set-theory-notes.pdf`, with particular attention to Sections V.5 and VIII.3.

THEOREM. *Let F be an ordered field satisfying the following conditions;*

(i) *If $a < b$ in F, then there is some rational number q such that $a < q < b$.*

(ii) *F is maximal in the sense that if E is an ordered field satisfying (i) and F is order-preservingly isomorphic to a subfield F_0 of E, then $E = F_0$.*

Then F is order-preservingly isomorphic to $\mathbb{R}$.

Sketch of proof. We shall use the *Dedekind cut construction* of the real numbers $\mathbb{R}$ from the rational numbers $\mathbb{Q}$. A Dedekind cut in $\mathbb{Q}$ is a pair of nonempty disjoint subsets $\{A, B\}$ such that

$A \cap B = \emptyset$ and $A \cup B = \mathbb{Q}$,

if $a_0 \in A$ and $a < a_0$ then $a \in A$, and if $b_0 \in A$ and $b > b_0$ then $b \in B$,

A has no largest element.

If $q \in \mathbb{Q}$ then q determines a Dedekind cut, with A equal to all rational numbers stricly less than q and B equal to all rational numbers greater than or equal to q. Irrational numbers like $\sqrt{2}$ determine Dedekind cuts similarly, but for an irrational number the set B does not have a least element.

This definition extends naturally to an arbitrary ordered field F satisfying (i) if we take A and B to be subsets of F which satisfy the properties listed above; let $\mathbb{D}(F)$ denote the set of Dedekind cuts in F.

The crucial observation is that one can imitate the construction of $\mathbb{R}$ from $\mathbb{Q}$ if the latter is replaced by an arbitrary ordered field; in particular, we can make $\mathbb{D}(F)$ into an ordered field containing F. This is a straightforward extension of the argument employed to construct $\mathbb{R} = \mathbb{D}(\mathbb{Q})$. Here is an online reference for that construction:

`http://www.uwec.edu/andersm/SETSVIII.pdf`

Furthermore, if F is an ordered field satisfying (i) and E is another ordered field with this property such that $F \subset E$, then the construction yields an order-preserving homomorphism of ordered fields from $\mathbb{D}(F)$ to $\mathbb{D}(E)$. The existence of the mapping requires some proof: Given a Dedekind cut $\{A, B\}$ in F, define A^* to be the set of all $x \in E$ such that $x \leq a$ for some $a \in F$, and define B^* to be the complement of A^* in B^*. All of the properties for Dedekind cuts except the second ones follow immediately, and the first part of the second property also follows immediately. To prove the second half of the second property, suppose that $y \in B^*$ and $y < x$, and assume that we do not have $x \in B^*$. Then we have $x \in A^*$, and therefore by the first half of the first property we also have $y \in A^*$, which is a contradiction. Therefore $\{A^*, B^*\}$ is a Dedekind cut in E. — The homomorphism and order preservation properties of the map $\mathbb{D}(F) \to \mathbb{D}(E)$ then follow from the construction. Since every homomorphism of fields is 1–1, it follows that all the field homomorphisms in this paragraph are 1–1.

Typo: Change B^* to E

The results of Section VIII.3 in `set-theory-notes.pdf` imply that every complete ordered field is uniquely isomorphic to $\mathbb{R}$, and hence if $F \subset E$ as in the preceding paragraph and we take the natural inclusion of $\mathbb{Q}$ in F, then the methods of the previously cited section show that we have isomorphisms of complete ordered fields $\mathbb{R} = \mathbb{D}(\mathbb{Q}) \to \mathbb{D}(F)$ and $\mathbb{D}(F) \to \mathbb{D}(E)$.

Suppose now that F satisfies (i) and the maximality condition in the theorem. Since F is an ordered subfield of $\mathbb{D}(F)$ it follows that $F = \mathbb{D}(F)$. If we combine this with the previously described isomorphism $\mathbb{R} \to \mathbb{D}(F)$, we obtain an order-preserving isomorphism between $\mathbb{R}$ and F.$\blacksquare$