AN EXAMPLE INVOLVING POWER SERIES

One can use either the Ratio Test or the Root Test to show that the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

converges. In fact, the convergence of this series and its value were known in the 14^{th} century. We shall describe how one can compute the value of this sum using power series. The original calculations of the sum used other methods.

We start with the geometric series

$$\sum_{n=0}^{\infty} x^n \quad = \quad \frac{1}{1-x}$$

which we know is valid for $|x| < 1$. We also know that we can differentiate power series term by term to obtain the derivative of the original infinite series. For the geometric series this yields the following formula:

$$\sum_{n=1}^{\infty} n\, x^{n-1} \quad = \quad \frac{1}{(1-x)^2}$$

If we multiply both sides by x we obtain the following equation, which by the termwise differentiation rule is valid for $|x| < 1$:

$$\sum_{n=1}^{\infty} n\, x^{n} \quad = \quad \frac{x}{(1-x)^2}$$

The series we want to compute is given by substituting $x = 1/2$ into the left hand side, so its value is given by substituting the same value into the right hand side. However evaluation of the right hand side at $x = 1/2$ yields a value of 2, and therefore we have computed the sum in question:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \quad = \quad 2$$