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Mathematics 153–001, Spring 2010, Examination 2

Answer Key

1. [*15 points*] If $\mathcal{L}$ is the secant line to the parabola $y = x^2$ which goes through the points $(1,1)$ and $(4,16)$, find the point (c, c^2) on the parabola at which the tangent line is parallel to $\mathcal{L}$. — Recall that two lines are parallel if their slopes are equal.

SOLUTION

The slope of the secant line is just

$$\frac{16-1}{4-1} \quad = \quad \frac{15}{3} \quad = \quad 5$$

while the slope of the tangent line at (c, c^2) is $2c$. The latter will be parallel to the secant line if and only if $2c = 5$, or equivalently if and only if $c = 5/2$.

2. [*20 points*] Suppose we are given two closed regions A and B in the first quadrant of the coordinate plane such that B is the result of moving A vertically upwards by k units. Suppose that the centroid of A has coordinates $(\overline{x}, \overline{y})$. Let S and T be the solids of revolution obtained by rotating A and B (respectively) about the x-axis. Using the Pappus Centroid Theorem, find the ratio Volume(T)/Volume(S). [*Hint:* What are the coordinates for the centroid of B?]

SOLUTION

The centroid of B is $(\overline{x}, \overline{y} + k)$. Therefore by the Pappus Centroid Theorem we have

$$\text{Volume}(S) \quad = \quad \text{Area}(A) \cdot 2\pi\overline{y} \ , \quad \text{Volume}(T) \quad = \quad \text{Area}(B) \cdot 2\pi(\overline{y} + k) \ .$$

Since B is obtained from A by vertical translation, the areas of B and A are equal. If we substitute this into the formula for the volume of T and take ratios, this implies that

$$\frac{\text{Volume}(T)}{\text{Volume}(S)} \quad = \quad \frac{\text{Area}(A) \cdot 2\pi(\overline{y} + k)}{\text{Area}(A) \cdot 2\pi\overline{y}} \quad = \quad \frac{\overline{y} + k}{\overline{y}} \ .$$

3. [*20 points*] Find all solutions of the equations $2x+3y+5z = 30$ and $x+y+z = 10$ for which x, y, z are all positive integers. There is at least one solution but the number of solutions is finite.

SOLUTION

If we multiply both sides of the second equation by 3 we get $3x + 3y + 3z = 30$, and if we subtract this from the first equation we obtain $2z - x = 0$. If we now substitute this last equation into the original second equation we obtain $y+3z = 10$, or $y = 10-3z$. Since we are looking for positive integer solutions, this means $z = 1, 2, 3$, so that $y = 7, 4, 1$ and thus

$$(x, y, z) \quad = \quad \text{one of} \quad (2, 7, 1)\ , (4, 4, 2)\ , (6, 1, 3)\ .$$

4. [*10 points*] The Greek trigonometric function $\mathrm{crd}\,\theta$ is given in modern terminology by $2\sin\frac{1}{2}\theta$. Find and derive a formula expressing $\mathrm{crd}\,(\pi-\theta)$ in terms of $\mathrm{crd}\,\theta$.

SOLUTION

We have $\mathrm{crd}\,(\pi-\theta) = 2\sin\frac{1}{2}(\pi-\theta)$, and the latter is just $2\sin\left(\frac{1}{2}\pi - \frac{1}{2}\theta\right)$, which is just $2\cos\frac{1}{2}\theta$. Since $\sin^2 + \cos^2 = 1$, this means that

$$\mathrm{crd}^{\,2}\theta \;+\; \mathrm{crd}^{\,2}(\pi-\theta) \quad=\quad 4\sin^2\tfrac{1}{2}\theta \;+\; 4\cos^2\tfrac{1}{2}\theta \quad=\quad 4\;.$$

This expresses $\mathrm{crd}\,(\pi-\theta)$ in terms of $\mathrm{crd}\,\theta$.

5. [*10 points*] For each of the following, state who or what came first. The first three are worth three points each, and the last three are worth two points each, so it is only necessary to answer two from each group correctly to earn full credit; extra credit will be given for additional correct answers.

(a) Al-Khwarizmi or Brahmagupta

(b) Alhazen or Omar Khayyam

(c) Oresme or Pacioli

(d) Mercator or Regiomontanus

(e) The Hindu-Arabic numeration system or the theory of geometrical perspective drawing.

(f) The basic results on the geometry of spherical triangles or the discovery of the reflection property for parabolas.

SOLUTION

(a) Brahmagupta (lived in the 7^{th} century, while Al-Khwarizmi lived in the 9^{th} century)

(b) Alhazen (lived mainly in the 11^{th} century, while Khayyam lived mainly in the 12^{th} century)

(c) Oresme (lived in the 14^{th} century, while Pacioli lived mainly in the 15^{th} century)

(d) Regiomontanus (lived in the 15^{th} century, while G. Mercator lived in the 16^{th} century)

(e) The Hindu-Arabic numeration system (was developed before the end of the 5^{th} century, while the theory of geometrical perspective drawing was developed during the 14^{th} to 16^{th} centuries)

(f) The discovery of the reflection property for parabolas (known to Greek mathematicians such as Apollonius in the 3^{rd} century B.C.E., while the basic results on the geometry of spherical triangles were discovered beginning in the 1^{st} century A.D. by Menelaus and others)

6. [*10 points*] Put the following figures from Greek mathematics in historical order.

[A] Claudius Ptolemy

[B] Eratosthenes

[C] Heron

[D] Pappus

[E] Proclus

[F] Plato

SOLUTION

F B C A D E

(These can be checked using `studyguide2.pdf` in the course directory.)

7. [*15 points*] Match any five of the following mathematicians with one item for which they are recognized. There may be several correct matches for a given name, but only one should be given (with zero credit if more than two are given). Extra credit will be granted if more than five names are correctly matched.

____ Al-Battani

____ Abu Kamil

____ Al-Karaji

____ Al-Kashi

____ Apollonius

____ Aryabhata

____ Bhaskara

____ Diophantus

____ Fibonacci

____ Hipparchus

____ Madhava

____ Menaechmus

____ Nasireddin

____ Oresme

____ Regiomontanus

A : Conic sections
B : Equation solving, algebraic formalism
C : Estimating π
D : Infinite series
E : Spherical geometry, trigonometry
F : Using forerunners of coordinates to describe points

(The solution is on the next page)

SOLUTION TO PROBLEM 7

In each case, it is enough to have one correct match.

Al-Battani E

Abu Kamil B

Al-Karaji B

Al-Kashi C E

Apollonius A C F

Aryabhata B C E

Bhaskara B C E

Diophantus B

Fibonacci B (partial credit for D — see below)

Hipparchus E

Madhava B C D E

Menaechmus A B

Nasireddin B E

Oresme B D F

Regiomontanus E

Note. The reason for partial credit is that the infinite sequence of Fibonacci numbers is sometimes described as a series. However, strictly speaking infinite sequences and infinite series are different types of objects in mathematics. An infinite sequence is an ordered list of terms, and an infinite series is the expression one obtains by trying to add the terms of an infinite sequence (the question of whether or not this makes sense is the issue of whether or not the series converges).