NAME: ______________________________

Mathematics 153, Spring 2020, Examination 2

INSTRUCTIONS: Work all questions, and unless indicated otherwise give reasons for your answers. The point values for individual problems are indicated in brackets. Please submit a copy of your exam by electronic mail, one to **EACH of the following THREE** addresses, by 11:59 P.M. on Tuesday, June 9, 2020:

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Each problem should be started on a separate sheet of paper. Make sure your submission is readable; some smartphone photos might not have enough contrast. Outside references such as course directory documents may be used, and you may discuss the problems informally with other students or ask for clarifications from either the teaching assistants or me, but the writeup you submit is required to be your own work.

The nominal top score for setting the curve will be 150 points.

1. [*20 points*] The following problem appears in Book II of Diophantus, *Arithmetica*:
Divide a positive number, say 20, into a sum of two positive rational numbers x, y such that for some positive rational number z both $z^2 + x$ and $z^2 + y$ are squares (of rational numbers). For the sake of definiteness, write $z^2 + x = (z + 2)^2$ and $z^2 + y = (z + 3)^2$.

2. [*20 points*] Let D be the region between the parabolas $y = x^2 + 1$ and $y = 3 - x^2$ where $|x| \leq 1$. Drawing a sketch is recommended.

(a) Find a horizontal line $y = C$ such that D is symmetric with respect to this line. You need to evaluate C explicitly, but you do not need to prove that your answer is the asserted value.

(b) Use Pappus' Centroid Theorem to find the volume for the solid of revolution obtained by rotating D around the x-axis.

3. [*20 points*] One number-theoretic result mentioned in the course was **Wilson's Theorem:** *If p is a prime then $(p-1)!$ is congruent to -1 mod p.* — The purpose of this exercise is to show the reverse implication.

(a) Suppose $n > 1$ is a composite integer ab where a and b are **unequal** integers both greater than 1. Prove that $(n-1)!$ is congruent to 0 mod n. [*Hint:* Why are both factors less than $n/2$?]

(b) The preceding part of the problem proves the reverse implication unless $n = p^2$ where p is a prime. Prove that if $p > 2$ is prime then $(p^2 - 1)!$ is congruent to 0 mod p^2, and find $k \in \{0, 1, 2, 3\}$ such that $(2^2 - 1)!$ is congruent to k mod 4.

4. [*20 points*] (a) The function $y(x) = x^3 + x$ is a strictly increasing function from the real line to itself which is 1–1 onto and hence has an inverse function. Use the Cubic Formula to write the inverse function $x(y)$.

(b) As noted in the Appendix at the end of `history09.pdf`, an infinite decimal expansion $0.x_1 x_2 x_3 \ldots$ represents a rational number if and only if it is eventually periodic; *i.e.*, one can find positive integers N and P such that $k \geq N$ implies $x_k = x_{k+P}$. Using this, prove that the real number

$$\sum_{m=1}^{\infty} 10^{-m^2}$$

is irrational. [*Hint:* Suppose it is, choose N and P as above, let $k \geq N$ so that the k^{th} term equals 1, and derive a contradiction.]

5. [*20 points*] Let C be a circle, and let P be a point not on the circle. Prove that the maximum and minimum distances from P to a point X on C occur when the line XP goes through the center of C. [*Hint:* Choose coordinate systems so that C is defined by $x^2 + y^2 = r^2$ and P is a point $(a, 0)$ on the x-axis with $a \neq \pm r$; use calculus to find the maximum and minimum for the square of the distance. Don't forget to pay attention to endpoints and places where a derivative might not exist.]

6. [*25 points*] This problem uses the conclusion of the previous one, so you may assume that conclusion here.

(a) Suppose that we are given two concentric circles with radii satisfying $0 < s < r$. Prove that the locus (= set) of all points which are equidistant from both circles is a third circle with the same center as the other two and radius equal to $\frac{1}{2}(9 + s)$.

(b) Now suppose we are given two circles with the same radius such that the distance between their centers is more than twice that radius. Prove that the locus (= set) of all points which are equidistant from both circles is the perpendicular bisector of the line joining their centers. [*Hint:* Why is the condition on the distances equivalent to saying that the distances between the point and the centers are equal?]

7. [*50 points*] In all cases, explanations for your answer may yield partial credit even if the answer itself is incorrect.

(*a*) Explain briefly why the increase in commerce during the later Middle Ages led to increased mathematical activity in Western Europe.

(*b*) Why were logarithms so useful for doing computations before the widespread use of electronic computers in the late 20^{th} century?

(*c*) Name one Arabic mathematician who discovered a geometrical fact which was apparently unknown to the Greeks.

(*d*) Name two ways in which the Indian concept of numbers during the first millenium A. D. was broader than the analogous Greek concept.

(*e*) Name one thing Fibonacci wrote about aside from the sequence of numbers which is now named after him.

(*f*) Put the following list of topics from calculus in historical order of study: Limits, Derivatives, Integrals, Infinite series