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Mathematics 153, Spring 2020, Examination 2

Answer Key

Solutions to (discarded) problems $4(b)$ and $6(b)$ from the first version of the exam are also given at the end of this file.

1. [*20 points*] The following problem appears in Book II of Diophantus, *Arithmetica*:

Divide a positive number, say 20, into a sum of two positive rational numbers x, y such that for some positive rational number z both $z^2 + x$ and $z^2 + y$ are squares (of rational numbers). For the sake of definiteness, write $z^2 + x = (z+2)^2$ and $z^2 + y = (z+3)^2$.

SOLUTION

The first sentence yields the equation $x + y = 20$, and as noted in the problem we are also assuming that $z^2 + x = (z+2)^2$ and $z^2 + y = (z+3)^2$. We need to solve for x, y and possibly z (the problem does not ask for the value, but we might as well do so).

If we add the third equation to the second and make the substitution $x + y = 20$, we obtain

$$2z^2 \; + \; 20 \quad = \quad 2z^2 \; + \; 10z \; + \; 13$$

which simplifies to $20 = 10z + 13$ and thus yields $z = 7/10$. Therefore it follows that

$$\frac{49}{100} \; + \; x \quad = \quad \left(\frac{27}{10}\right)^2 \qquad \text{so that}$$

$$x \quad = \quad \frac{729 - 49 = 680}{100} \quad = \quad \frac{34}{5} \qquad \text{and} \qquad y \quad = \quad 20 \; - \; \frac{34}{5} \quad = \quad \frac{66}{5} \; .$$

One way to check this result is to verify that $z^2 + y = (z+3)^2$ if y and z are given as above.■

<u>NOTE:</u> One can always find solutions to this type of equation if 2, 3 and 20 are replaced by a, b, c such that $a^2 + b^2 < c$.

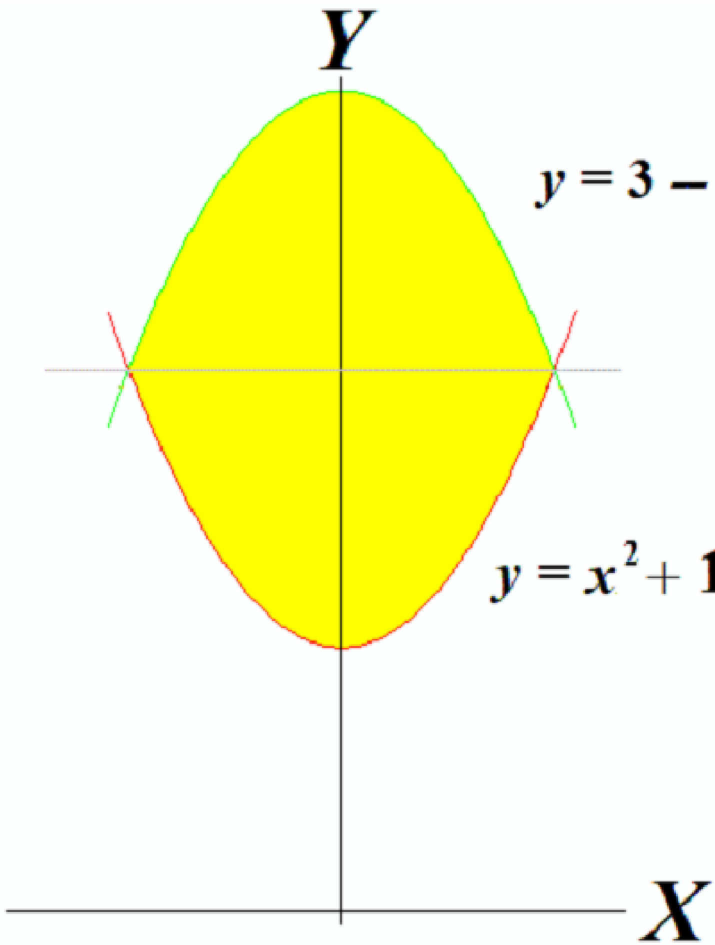
2. [*20 points*] Let D be the region between the parabolas $y = x^2+1$ and $y = 3-x^2$ where $|x| \leq 1$. Drawing a sketch is recommended.

(a) Find a horizontal line $y = C$ such that D is symmetric with respect to this line. You need to evaluate C explicitly, but you do not need to prove that your answer is the asserted value.

(b) Use Pappus' Centroid Theorem to find the volume for the solid of revolution obtained by rotating D around the x-axis.

SOLUTION

(a) **Acceptable geometric solution.** If we graph the two parabolas we see that they meet at the points $(\pm 1, 2)$, and the figure is symmetric with respect to the horizontal line $y = 2$.



Algebraic solution. The mirror image of a point (x, y) with respect to the line $y = C$ is $(x, y) \to (X, Y) = (x, 2C - y)$. Therefore we need to choose C so that the upper curve $Y = 3 - X^2$ is the mirror image of the lower curve $y = x^2 + 1$; specifically, we want

$$3 \ - \ x^2 \ = 2C \ - \ (x^2 + 1)$$

from which we conclude that $C = 2$.

(b) The preceding discussion implies that the centroid has coordinates $(a, 2)$, where we know the second coordinate by symmetry considerations and we don't need to evaluate a (however, the figure is also symmetric with respect to the y-axis so $a = 0$). Now the area of the region is equal to

$$2 \cdot \int_{-1}^{1} (1 - x^2)\, dx \quad = \quad 2 \cdot \left[x - \frac{x^3}{3} \right]_{-1}^{1} \quad = \quad \frac{8}{3}$$

and therefore the Pappus Centroid Theorem for Volumes implies that the volume for the solid of revolution is equal to

$$2\pi \cdot 2 \cdot \frac{8}{3} \quad = \quad \frac{32\pi}{3}$$

where the second factor on the left hand side appears because the y-coordinate of the centroid is 2.■

3. [*20 points*] One number-theoretic result mentioned in the course was **Wilson's Theorem:** *If p is a prime then $(p-1)!$ is congruent to -1 mod p.* — The purpose of this exercise is to show the reverse implication.

(a) Suppose $n > 1$ is a composite integer ab where a and b are **unequal** integers both greater than 1. Prove that $(n-1)!$ is congruent to 0 mod n. [*Hint:* Why are both factors less than or equal to $n/2$?]

(b) The preceding part of the problem proves the reverse implication unless $n = p^2$ where p is a prime. Prove that if $p > 2$ is prime then $(p^2 - 1)!$ is congruent to 0 mod p^2, and find $k \in \{0, 1, 2, 3\}$ such that $(2^2 - 1)!$ is congruent to k mod 4.

SOLUTION

(a) Both of the unequal numbers a and b are between a and $n-1$; in fact they satisfy $1 > a, b, < \frac{1}{2}n$. Therefore both a and b are factors appearing on the right hand side of $(n-1)! = 1 \times 2 \times \cdots \times (n-1)$. But this means that ab evenly divides the factorial, and since $n = ab$ it follows that $(n-1)!$ is congruent to 0 mod n.■

(b) Since p is an odd prime, both p and $2p$ are less than p, so as before we know that $2p^2$ evenly divides $(p^2 - 1)!$, so that $(p^2 - 1)!$ is congruent to 0 mod p^2. Finally, if $p = 2$ then $6 = 3!$ is congruent to 2 mod 4.■

4. [*20 points*] (a) The function $y(x) = x^3 + x$ is a strictly increasing function from the real line to itself which is 1–1 onto and hence has an inverse function. Use the Cubic Formula to write the inverse function $x(y)$.

(b) Find a degree 4 polynomial $p(x)$ with integral coefficients such that

$$\sqrt{1+\sqrt{5}}$$

is a root of $p(x)$.

SOLUTION

(a) The problem requires that we find the root(s) of the equation $x^3 + x = y$ as a function of y. The Cubic Formula states that a solution to $x^3 + px = q$ is given by

$$x \quad = \quad \sqrt[3]{\sqrt{(p/3)^3 + (q/2)^2} + (q/2)} \;-\; \sqrt[3]{\sqrt{(p/3)^3 + (q/2)^2} - (q/2)}$$

and for our example $p = 1$ and $q = y$. Therefore the formula for the inverse function is

$$x \quad = \quad \sqrt[3]{\sqrt{(1/3)^3 + (y/2)^2} + (y/2)} \;-\; \sqrt[3]{\sqrt{(1/3)^3 + (y/2)^2} - (y/2)} \;.\blacksquare$$

(b) If $x = \sqrt{1+\sqrt{5}}$, then $x^2 = 1+\sqrt{5}$, so $x^2 - 1 = \sqrt{5}$ and hence $(x^2-1)^2 = 5$. This can be rewritten as the quartic equation

$$x^4 \;-\; 2x^2 \;+\; 1 \quad = \quad 5 \qquad \text{or equivalently}$$

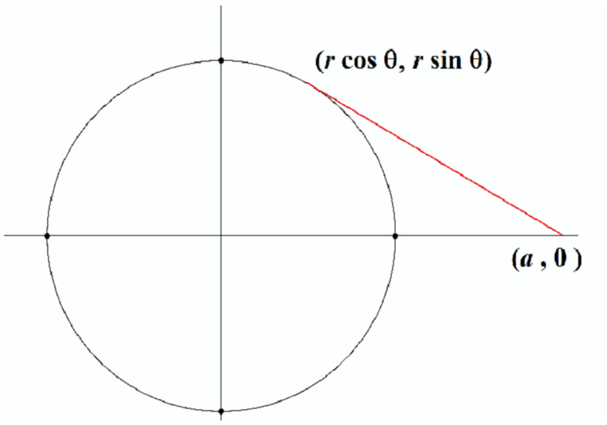
$x^4 \;-\; 2x^2 \;-\; 4 \quad = \quad 0.\blacksquare$

5. [*20 points*] Let C be a circle, and let P be a point not on the circle. Prove that the maximum and minimum distances from P to a point X on C occur when the line XP goes through the center of C. [*Hint:* Choose coordinate systems so that C is defined by $x^2 + y^2 = r^2$ and P is a point $(a, 0)$ on the x-axis with $a \neq \pm r$; use calculus to find the maximum and minimum for the square of the distance. Don't forget to pay attention to endpoints and places where a derivative might not exist.]

SOLUTION

Take a coordinate system described in the hint; such a system exists by the result in `history11c.pdf`. The points on C are those whose rectangular coordinates have the form $(r\cos\theta, r\sin\theta)$ where $\theta \in [0, 2\pi]$, and the square of the distance from such a point to P is given by

$$d^2(\theta) \quad = \quad (a - r\cos\theta)^2 \; + \; r^2\sin^2\theta \quad = \quad a^2 \; + \; r^2 \; - \; 2ar\cos\theta \; .$$



We need to find the maximum and minimum values for this function.

The maximum and minimum values of d^2 can be found by evaluating this function at the end points and the points where the derivative is zero; since the displayed formula gives a function which is differentiable everywhere, we need not worry about points where the derivative may be undefined (there are none). At the end points where $\theta = 0$ or 2π, the value of the function is $a^2 + r^2 - 2ar = (a - r)^2$. Also, if we differentiate the square of the distance with respect to theta we obtain the formula $D\left(d^2(\theta)\right) = 2ar\sin\theta$, which is zero at the end points and also when $\theta = \pi$. At the latter point the value of d^2 is equal to $a^2 + r^2 + 2ar = (a + r)^2$.

If we translate this into geometry, we see that the closest point corresponds to $\theta = 0$ and hence this point has coordinates $(|a - r|, 0)$. Similarly, the farthest point corresponds to $\theta = \pi$ and hence this point has coordinates $(|a + r|, 0)$. Therefore the maximum and minimum distances occur for points on the x-axis, which is the line joining P to the center of the circle.

Note that if $a > 0$ then the latter is just $(a+r, 0)$, and the closest point has coordinates $(a - r, 0)$ if $a > r$ and $(r - a, 0)$ if $a < r$.■

6. [*25 points*] This problem uses the conclusion of the previous one, so you may assume that conclusion here.

Suppose that we are given two concentric circles with radii satisfying $0 < s < r$. Prove that the locus (= set) of all points which are equidistant from both circles is a third circle with the same center as the other two and radius equal to $\frac{1}{2}(r+s)$.

See the next page for a drawing.

Here is a list of suggested steps for solving the first part. If a point is on either circle, then it cannot be equidistant from the two circles (the distance to one is zero, the distance to the other is positive), so let's assume we are looking at points which are on neither circle. The **first part** is to show that if a point is equidistant, then it lies on the circle.

(0) Choose a coordinate system so that $\mathbf{0} = (0,0)$ is the center of both circles.

(1) Why are polar coordinates (ρ, θ) for points on the smaller circle given by all ordered pairs (s, θ) where θ runs through all real numbers and similarly the points on the larger circle given by all ordered pairs (r, θ) where θ runs trough all real numbers?

(2) Given a point P with polar coordinates (u, α) where $u > 0$, why does the preceding exercise imply that the points on the circle which are closest to P have polar coordinates (s, α) and (t, α), and what does this imply for the distances between P and the two circles? Note that there are several cases depending upon which of the statements $0 < u < s < t$, $0 < s < u < t$ or $0 < s < t < u$ is true.

(3) Using the preceding division into cases, prove that if P is equidistant from the two circles, then $0 < s < u < t$ and in fact $u = \frac{1}{2}(t+s)$. [*Hint:* What is the distance between two points with polar coordinates (X, α) and (Y, α) where the second coordinates are equal and the first coordinates are positive?]

(4) Why does this conclude the first half of the proof?

The **second part** is to show that if a point P lies on the given circle of radius $\frac{1}{2}(r+s)$, then it is equidistant from the original two circles.

(5) Show that the distance between $P = \left(\frac{1}{2}(r+s), \alpha\right)$ and each of the two circles is equal to $\frac{1}{2}(r-s)$. [*Hint:* What points on the two circles are closest to P? Why do we know this is true?]

SOLUTION

We shall give the reasons for each of the steps described above.

(0) By `history11c.pdf` we can always choose a coordinate system so that at given point corresponds to the origin.

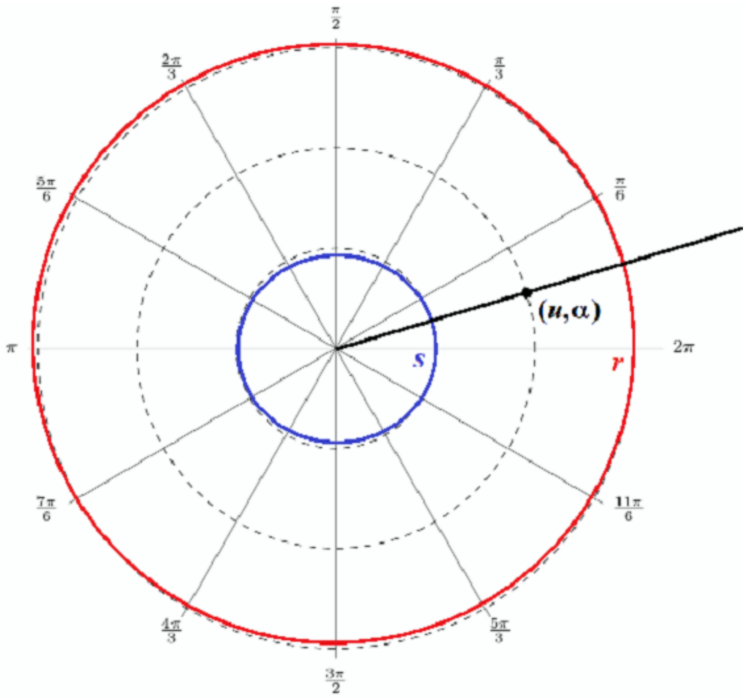
(1) These are true by the definition of polar coordinates and we have circles whose centers are the origin and whose radii are t and s.

(2) By Exercise 5 the closest point to P is a point on the line joining P to the origin and this line is definable by the equation $\theta = \alpha$. The distances between P and these points depend upon where u lies in the positive real numbers. If $u = s$ or $u = t$, then the distance to one circle is 0 and the distance to the other circle is $t - s > 0$. If $u < s < t$ then the distances are $s - u$ and $s - t$; note that the first is less than the second. If $s < r < u$ then the distances are $u - s$ and $t - u$. Finally, if $s < t < u$ then the distances are $u - t$ and $u - s$; note that the first is now greater than the second.

(3) The only case in (2) for which the distances might be equal is the case $s < u < r$, in which case the point is equidistant from the circles if and only if $u - s = t - u$. The latter equation is equivalent to $u = \frac{1}{2}(t + s)$.

(4) By the previous step we know that if the point P with polar coordinates (u, α) is equidistant from the two circles then it has polar coordinates $\left(\frac{1}{2}(t + s), \alpha\right)$ and hence lies on the circle centered at the origin with radius $\frac{1}{2}(t + s)$.■

(5) Conversely, if P has polar coordinates $\left(\frac{1}{2}(t + s), \alpha\right)$ then the first coordinate is in the open interval (s, t), and by (2) the distances from P to these circles are given by $\frac{1}{2}(t + s) - s$ for the first circle and $t - \frac{1}{2}(t + s)$ for the second. Since both of these expressions simplify to $\frac{1}{2}(t + s)$, the point P is equidistant from both circles.■



7. [*50 points*] In all cases, explanations for your answer may yield partial credit even if the answer itself is incorrect.

(a) Explain briefly why the increase in commerce during the later Middle Ages led to increased mathematical activity in Western Europe.

(b) Why were logarithms so useful for doing computations before the widespread use of electronic computers in the late 20^{th} century?

(c) Name one Arabic mathematician who discovered a geometrical fact which was apparently unknown to the Greeks.

(d) Name two ways in which the Indian concept of numbers during the first millenium A. D. was broader than the analogous Greek concept.

(e) Name one thing Fibonacci wrote about aside from the sequence of numbers which is now named after him.

(f) Put the following list of topics from calculus in historical order of study: Limits, Derivatives, Integrals, Infinite series

SOLUTIONS

(a) Increased trade required more complicated arithmetic. Merchants needed to learn how to carry out the more complex computations, instructors needed to know enough to teach them, and in some cases the instructors began to study algebraic questions of independent interest.■

(b) They allowed the users of mathematics to convert multiplication problems to addition problems and vice versa, which was useful because adding numbers with many significant digits is generally easier than multiplying such numbers.■

(c) Possible names include Thabit ibn Qurra, Abul-Wafa, Nasreddin, Omar Khayyam and potentially others (with suitable supporting evidence).■

Acceptance of zero is also correct.

(d) Three ways in which Indian mathematics had a broader concept of numbers were the recognition of negative numbers, the equal status of rational and irrational numbers, and attempts to understand infinite numbers (only two were needed).■

(e) There are many possibilities including an influential description of the Hindu-Arabic numeration system, perfect squares in an arithmetic progression, solving a cubic equation, and potentially others (with suitable supporting evidence).■

(f) Integrals, Infinite series, Derivatives, Limits. One can give a credible argument that the first two should be reversed, and therefore an answer with these interchanged (and the first two correctly ordered) is worth full credit.■

ORIGINAL 4(b). As noted in the Appendix at the end of `history09.pdf`, an infinite decimal expansion $0.x_1x_2x_3$ ... represents a rational number if and only if it is eventually periodic; *i.e.*, one can find positive integers N and P such that $k \geq N$ implies $x_k = x_{k+P}$. Using this, prove that the real number

$$\sum_{m=1}^{\infty} 10^{-m^2}$$

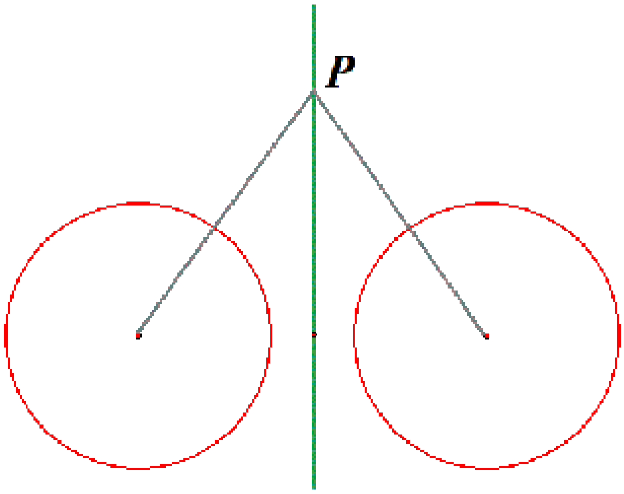
is irrational. [*Hint:* Suppose it is rational, choose N and P as above, let $k \geq N$ so that the k^{th} term equals 1, and derive a contradiction.]

SOLUTION

Follow the hint, so that k must be m^2 for some m. Suppose that the period P satisfies $P \leq 2m$. Then the next nonzero term should be a 1 in place $m^2+P \leq m^2+2m < (m+1)^2$. Since the next nonzero term occurs in place $(m+1)^2$, this cannot happen.

It will suffice to show that $P \geq 2m+1$ is also impossible. If this were true, then for each nonnegative integer r we also have $1 = x_{m^2+rP}$. Therefore if x_a and x_b are in the decimal expansion of x such that $a < b$ and there are no nonzero terms x_t such that $a < t < b$, then we must have $b - a \leq P$. By construction we know that there are no nonzero terms between the q^2 and $(q+1)^2$ places, and consequently we must have $2q+1 = (q+1)^2 - q^2 \leq P$. Since q is an arbitrary large integer, it follows that $P \geq 2q+1$ for all sufficiently large q. There is no positive integer P which can satisfy this condition, and this means that the decimal expansion cannot be eventually periodic. It follows that the infinite sum in the problem cannot be a rational number.■

ORIGINAL 6(b). Suppose we are given two circles with equal radii such that the distance between their centers is more than twice that radius. Prove that the locus (= set) of all points which are equidistant from both circles is the perpendicular bisector of the line joining their centers. [*Hint:* Why is the condition on the distances equivalent to saying that the distances between the point and the centers are equal?]



SOLUTION

We first note that if d is the common distance from the point to the two centers and r is the radii of the circles, then $d > r$. To see this, observe that if P is the point and A, B are the circle centers then by the Triangle Inequality we have

$$2r \quad < \quad \text{distance}\,(A, B) \quad \leq \quad \text{distance}\,(A, P) \;+\; \text{distance}\,(B, P) \quad = \quad 2d\ .$$

Now choose a coordinate system such that A is the origin and B has coordinates $(c, 0)$ where $c > 0$ is the distance between A and B.

(a) Suppose first that P is equidistant from the two circles, and let C, D be the points at which the distance to P is minimized. Then by Exercise 5 we know that the centers A and B lie on the respective lines from P to C and D. Furthermore, we know that C and D lie on the open segments (AP) and (BP) respectively. Since d is the length of the two segments $[CP]$ and $[DP]$, it follows that $d + r$ is the length of the two segments $[AP]$ and $[BP]$. Therefore the point P is equidistant from A and B, which means that P lies on the perpendicular bisector of the segment $[AB]$.■

(b) Suppose now that P lies on the perpendicular bisector of the segment $[AB]$. The by the locus theorem for perpendicular bisectors we know that the lengths of $[AP]$ and $[BP]$ are equal. Since the distance from A to B is greater than $2r$, it follows (say, from the Pythagorean Theorem) that the common length of $[AP]$ and $[BP]$ is greater than r. Let $C \in (AP)$ and $D \in (BP)$ be points such that the lengths of $[AC]$ and $[BD]$ are both equal to r. Then by Exercise 5 the minimum distances from P to the circles are the lengths of $[PC]$ and $[PD]$; these lengths are equal to

$$\text{distance}\,(A,P)\ -\ \text{distance}\,(A,C) \quad \text{and} \quad \text{distance}\,(B,P)\ -\ \text{distance}\,(B,D)\ .$$

By assumption the first terms in both sums are equal, and the second terms are equal because they are both equal to r. Therefore the point P is equidistant from the two circles.■