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Mathematics 153, Spring 2019, Examination 3

Answer Key

(*Modification to Problem 2 inserted*)

1. [*25 points*] (a) Find a finite continued fraction expression for 2/9.

(b) For which of $n = 6, 10, 14, 15$ is $2^n \equiv 2 \bmod n$? [*Hint:* If $x^a \equiv y\ (n)$ then $x^{ab} \equiv y^b\ (n)$.]

SOLUTION

(a) We have

$$\frac{2}{9} = \frac{1}{\frac{9}{2}} = \frac{1}{4+\frac{1}{2}}. \quad \blacksquare$$

(b) If $n = 6$ then $2^6 = 64 \equiv 4\ (6)$. NO FOR 6.

If $n = 10$ then $2^5 = 32 \equiv 2\ (10)$, and hence $2^{10} = \left(2^5\right)^2 \equiv 2^2 = 4 \equiv 4\ (10)$. NO FOR 10.

If $n = 14$ then $2^7 = 128 \equiv 2\ (14)$, and hence $2^{14} = \left(2^7\right)^2 \equiv 2^2 = 4 \equiv 4\ (14)$. NO FOR 14.

If $n = 15$ then $2^7 = 128 \equiv 8\ (15)$, and hence $2^{15} = 2 \cdot \left(2^7\right)^2 \equiv 2 \cdot 8^2 \equiv 2 \cdot 4 \equiv 8\ (15)$. NO FOR 15.

2. [*20 points*] Write the number

$$\sqrt[3]{-5+\sqrt{-2}}$$

in the form $a+b\sqrt{-2}$, where a and b are single digit integers. [*Hints:* If we have an equation of the form $xy = \pm k$ where x and y are integers and either k is a prime or $k=1$, then we know that one of x, y is $\pm k$ and the other is ± 1. In the problem there are expressions of this type where x and y are polynomials in a and b.]

SOLUTION

Following Bombelli's method we want to find a and b so that

$$-5 \ + \ \sqrt{-2} \quad = \quad \left(a \ + \ b\sqrt{-2}\right)^3 \ .$$

Expand the right hand side using the Binomial Formula and equate the integral parts and the coefficients of $\sqrt{-2}$:

$$\left(a+b\sqrt{-2}\right)^3 \quad = \quad (a^3-6ab^2) \ + \ (3a^2b-2b^3)\sqrt{-2}$$

Hence we have $a^3-6ab^2=-5$ and $3a^2b-2b^3=1$, where a and b are supposed to be single digit integers. Observe that we have factorizations

$$-5 \quad = \quad a(a^2-6b^2)\ , \qquad 1 \quad = \quad b(3a^2-2b^2)$$

and that we can apply the hint to each of these. In the second equation we obtain $b = \pm 1 = 3a^2-2b^2 = 3a^2-2$. The only possible solutions to the second equation are $a = \pm 1$, and this implies that $b = 3-2 = 1$.

Now consider the first equation. It must have the form $-5 = a(1-6) = -5a$ which means that $a=1$. Therefore the desired cube root is equal to $1+\sqrt{-2}$.∎

3. [*25 points*] Consider the two points $p = (0, 0)$ and $q = (4a, 0)$ (where $a > 0$) and let F be the locus (or set) of all points $z = (x, y)$ such that the distance from z to q is 3 times the distance from z to p. Is F a conic section curve? If so, describe it. [*Hint:* What can we say about the squares of the distances?]

SOLUTION

The hint tells us that we may rephrase the problem so that the square of the distance from z to q is 9 times the square of the distance from z to p. We shall use this version henceforth.

Writing things out in coordinates we have

$$(x - 4a)^2 + y^2 \quad = \quad 9 \cdot \left(x^2 + y^2\right)$$

which simplifies to

$$8x^2 + 8y^2 \quad = \quad 16a^2 - 8ax \ , \qquad \text{or equivalently} \quad x^2 + ax + y^2 = 2a^2$$

and after completing the square the latter is equivalent to

$$\left(x + \frac{a}{2}\right)^2 \ + \ y^2 \quad = \quad \frac{9a^2}{4}$$

which is clearly the equation of a circle; specifically, the center is $\left(-\frac{1}{2}\,a, 0\right)$ and the radius is $\frac{3}{2}\,a$.■

4. [*30 points*] Answer each of the following questions. If your answers are incorrect but supporting reasons are included, there is a chance of partial credit.

(a) Assuming that the Fibonacci sequence starts with $F_1 = F_2 = 1$, give a definition of F_n when $n \geq 2$ in terms of F_m for selected values of $m < n$.

ANSWER

$F_n \quad = \quad F_{n-1} \ + \ F_{n-2}$.■

(b) For each of the pairs of names below, determine who came first. Five correct answers will earn full credit, and additional correct responses will earn extra credit.

ANSWERS APPEAR AFTER THE RESPECTIVE PAIRS.

Cardano or Descartes.

Cardano. Cardano was active in the 16^{th} century, Descartes mainly in the 17^{th} century.

al-Khwarizmi or Aryabhata.

Aryabhata. Aryabhata was active in the $5^{\text{th}} - 6^{\text{th}}$ century, al-Khwarizmi in the 9^{th} century.

Barrow or Leibniz.

Barrow. Barrow was active in the early to middle 17^{th} century, Leibniz mainly in the late 17^{th} and early 18^{th}century.

Euler or Weierstrass.

Euler. Euler was active in the 18^{th} century (but his last published paper appeared in 1831), Weierstrass was active in the 19^{th} century.

Bolzano or Dedekind.

Bolzano. Bolzano was active mainly in the early 19^{th} century , Dedekind mainly later in that century.

Gregory or Robinson.

Gregory. Gregory was active in the 17^{th} century, Robinson in the 20^{th} century.

Pascal or Viète.

Viète. Viète was active in the early 16^{th} and the early 17^{th} century , Pascal later in the 17^{th} century.

Oresme or Regiomontanus.

Oresme. Oresme was active in the 14^{th} century, Regiomontanus in the 15^{th} century.