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# Mathematics 153, Spring 2021, Examination 1

## Answer Key

**1.** [*25 points*] Let $p$ and $q$ be odd primes. Prove that $p^2q$ is not a perfect number. [*Hint:* Factor $1+p+q+pq$ as a product of two binomials.]

SOLUTION

The proper divisors of $p^2q$ are 1, $p$, $q$, $pq$ and $p^2$, so their sum is

$$1 + p + q + pq + p^2 = (1+p)(1+q) + p^2 .$$

If $p^2q$ were a perfect number this would be equal to the displayed expression, so we need to show this is impossible. Note that $p, q \geq 3$.

Here is one way of obtaining a contradiction: If we did have $(1+p)(1+q)+p^2 = p^2q$ then we could rewrite this as $(1+p)(1+q) = p^2(q-1)$, so that

$$\frac{q-1}{q+1} = \frac{p+1}{p^2} .$$

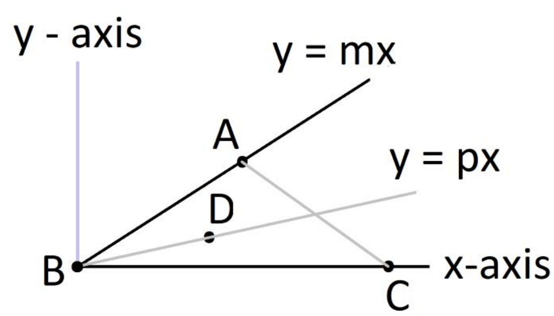
But $p, q \geq 3$ implies that the left hand side is always greater than or equal to $\frac{1}{2}$ and the right hand side is always less than or equal to $\frac{4}{9}$, which is strictly less than $\frac{1}{2}$. Thus we have a contradiction, and its source was the assumption that $p^2q$ is a perfect number. Therefore $p^2q$ is not a perfect number.■

**2.** [*25 points*] Using coordinate geometry, prove the following case of the **Crossbar Theorem**, which is one result tacitly assumed in Euclid's *Elements* but not stated explicitly: Let $m > 0$ and consider $\angle ABC$ in the coordinate plane $\mathbb{R}^2$ where $A = (u, mu)$, $B = (0,0)$ and $C = (v, 0)$ where $v > u > 0$. Let $D = (1, p)$ where $0 < p < m$, so that $D$ lies in the interior of $\angle ABC$. Prove that the lines $BD$ and $AC$ meet at a point $(x, y)$ such that $v > x > u$. [*Hint:* Draw a picture to make the problem more transparent.]

SOLUTION

We shall follow the additional hint that was posted:

> Let $Y = g(X)$ give the linear function defining the line $AC$, and let $f(X,Y) = Y - pX$, so that the line $BD$ is defined by $f(X,Y) = 0$. Now let $h(X) = f(X, g(X))$. This is a continuous function (you need not verify this). $(a)$ Why does $h(X) = 0$ mean that $(X, g(X))$ lies on both $BD$ and $AC$? $(b)$ What are the signs of $h(u)$ and $h(v)$, and why does this imply that $h(x) = 0$ for some $x$ between $u$ and $v$?




The reason for $(a)$ is that $0 = h(X) = f(X, g(X))$ translates to $0 = Y - pX$ if we have $Y = g(X)$. So $0 = h(X)$ means that there is a point $E$ on $AC$ — namely, $(X, g(X))$ — such that $E$ also lies on the line $BD$.

The first parts of $(b)$ follow by direct calculation: By construction $g(u) = mu$ and hence $h(u) = f(u, mu) = mu - pu$, and this is positive because $m > p$ and $u > 0$. On the other hand, we also have $g(v) = 0$ by construction, and therefore $h(v) = f(v, 0) = 0 - pv$, and this is negative because $v, p > 0$. Since the function $h$ is continuous, the Intermediate Value Theorem from calculus implies that $h(x) = 0$ for some $x$ such that $u < x < v$, and by $(a)$ this means that the point $(x, px)$, which lies on $BD$ by construction, must also lie on $AC$.$\blacksquare$

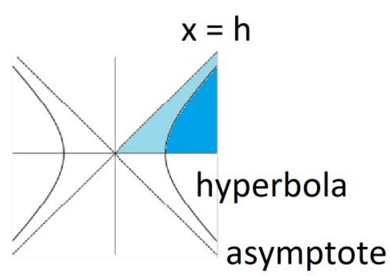
**3.** [*25 points*] The following problem is similar in spirit to some which were studied by Archimedes and others. Solve it using integral calculus: Let $A_h$ be the closed region in the coordinate plane defined by the vertical lines $1 = x$ and $x = h$ (where $h > 1$), the $x$-axis, and the hyperbola $y = \sqrt{x^2 - 1}$, and let $B_h$ be the corresponding region defined by the vertical lines $0 = x$ and $x = h$ (where $h > 0$), the $x$-axis, and the hyperbola's asymptote $y = x$. Next, let $P_h$ and $Q_h$ be the solids of revolution obtained by rotating $A_h$ and $B_h$ (respectively) about the $x$-axis. Compute the ratio

$$\frac{\mathrm{Vol}\,(P_h)}{\mathrm{Vol}\,(Q_h)} \ .$$

[*Hint:* Draw a picture to make the problem more transparent.]

SOLUTION

In the drawing below, the region $A_h$ is the union of the regions colored in medium and light blue, and $B_h$ is the region colored in medium blue.




By the standard Disk Method formula for the volume of a solid of revolution about the $x$-axis, we have

$$\mathrm{Vol}\,(P_h) \quad = \quad \pi \cdot \int_1^h x^2 - 1 \; dx \ , \qquad \mathrm{Vol}\,(Q_h) \quad = \quad \pi \cdot \int_0^h x^2 \; dx$$

and hence we have

$$\mathrm{Vol}\,(P_h) \quad = \quad \pi \cdot \left( \frac{x^3}{3} \ - \ x \right) \Bigg|_1^h \ , \qquad \mathrm{Vol}\,(Q_h) \quad = \quad \pi \cdot \frac{x^3}{3} \Bigg|_0^h \ .$$

If we evaluate these expressions we find that

$$\mathrm{Vol}\,(P_h) \quad = \quad \pi \left( \frac{h^3}{3} \ - \ h \ + \ \frac{2}{3} \right) \ , \qquad \mathrm{Vol}\,(Q_h) \quad = \quad \frac{\pi\, h^3}{3} \ .$$

It follows that

$$\frac{\mathrm{Vol}\,(P_h)}{\mathrm{Vol}\,(Q_h)} \quad = \quad \frac{h^3 - 3h + 2}{h^3} \ .\blacksquare$$

**4.** [*25 points*] For each of the following, state whether the following was apparently first known to, or described by, Greek mathematicians during the Ionic, Athenian or Hellenistic period and give reasons for your answer.

($a$) The volume ratio of a cone to a cylinder, where both have the same base and height.

($b$) Recognition of the semiregular polyhedron whose faces resemble the pattern if a modern soccer ball.

($c$) In a right triangle, the midpoint of the hypotenuse is equidistant from all three vertices. [*Hint:* Is there a circle containing all three vertices?]

($d$) Precise description of the logical setting for mathematics.

SOLUTIONS

($a$) ATHENIAN. This result is attributed to Democritus (460 BCE – 370 BCE, better known for his atomic theory of matter).■

($b$) HELENISTIC. The discovery of this polyhedron is attributed to Archimedes (287 BCE – 212 BCE).■

($c$) IONIC. This is essentially part of a result attributed to Thales (624 BCE – 548 BCE) about angles inscribed in semicircles.■

($d$) ATHENIAN. The definitive framework for Greek mathematics (still valid today in many respects) was due to Plato (428 BCE – 348 BCE) and Aristotle (384 BCE – 322 BCE).■