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Mathematics 153, Spring 2021, Examination 2

Answer Key

1. [*25 points*] Given a positive integer x, let $f(x)$ denote the following partial fraction expression:

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{x}}}$$

Show that if $x < y$ then $f(x) > f(y)$. [*Hints:* Simplify the right hand side into a quotient of two polynomials, and recall that if A, B, C, D are positive real numbers, then $A/B < C/D$ if and only if $AD < BC$.]

SOLUTION

Simplify the continued fraction from the inside out. The first step is to simplify $3 + (1/x)$, which is $(3x+1)/x$ and whose reciprocal is $x/(3x+1)$. The next step is to rewrite

$$2 + \frac{x}{3x+1} = \frac{7x+2}{3x+1}$$

and take is reciprocal, which is $(3x+1)/(7x+2)$. Finally, we rewrite

$$1 + \frac{3x+1}{7x+2} = \frac{10x+3}{7x+2}$$

and recall that we want to show this is strictly decreasing as a function of x, at least if we take x to be a positive integer.

This is where the second hint is needed. We want to show that $x < y$ implies that

$$\frac{10x+3}{7x+7} > \frac{10y+3}{7y+2} .$$

By the hint this is equivalent to verifying that

$$(10x+3)\cdot(7y+2) = 70xy+20x+21y+6 > 70xy+21x+20y+6 = (10y+3)\cdot(7x+2)$$

and if we subtract the common summand $70xy + 20x + 20y + 6$ from each side we are left with the equivalent statement $y > x$, which by the argument is true if and only if $f(x) > f(y)$. Since we assumed that $x < y$, it follows that $f(x) > f(y)$ is also true.■

Alternative approach: One can also prove this using the formula for *f*(*x*) as a quotient of two linear polynomials and showing that the derivative is negative for *x* > 1.

2. [*25 points*] (a) Let p be a prime. Prove that if the integers a, b satisfy $a \equiv b \bmod p$ then $a^p \equiv b^p \bmod p^2$. [*Hint:* Look at the proof of the Little Fermat Theorem.]

(b) Let Give an example of an even positive integer n and an integer a such that $a^n \not\equiv a \bmod n$. By the Little Fermat Theorem we must have $n > 2$.

SOLUTION

(a) Since $a \equiv b \bmod p$, we have $b = a + kp$ for some integer k. Therefore

$$b^p \quad = \quad (a+kp)^p \quad = \quad \sum_{r=0}^{p} \binom{p}{r} a^{n-r} \, k^r \, p^r \ .$$

If $r > 0$ then p clearly divides p^r, and the argument on page 514 of Burton (recall the hint!) implies that the prime p also divides the binomial coefficient

$$\binom{p}{r} \quad = \quad \frac{p!}{(p-r)!\, r!}$$

and therefore p^2 divides the product

$$\binom{p}{r} a^r k^r p^r$$

for every $r > 0$. But this means that $b^p \equiv a^p \bmod p^2$.∎

(b) Let $n > 2$ be an arbitrary even number, and let $a = -1$. Then $-1 \not\equiv 1 \bmod n$, but if $n = 2k$ then $a^n = (-1)^{2k} = ((-1)^2)^k = 1^k = 1 \not\equiv -1 = a$. Thus we have examples for every even $n > 2$.∎

3. [*25 points*] (a) The function $y(x) = 2x^3 + 3x$ is a strictly increasing function from the nonnegative real numbers to itself which is 1–1 onto, and hence it has an inverse function. Use the Cubic Formula to give an explicit formula the inverse function $x(y)$ involving the four basic arithmetic operations and taking square and cube roots.

(b) It is known that the integral function

$$\int_0^x \frac{dt}{\sqrt{1-t^4}}$$

cannot be expressed in terms of the functions which arise in first year calculus. Using Newton's Binomial Series find the first four nontrivial terms of an infinite series for this function involving powers of x (*i.e.*, a partial Maclaurin series).

SOLUTION

(a) The problem requires that we find the root(s) of the equation $2x^3 + 3x = y$ as a function of y. The Cubic Formula states that a solution to $x^3 + px = q$ is given by

$$x \quad = \quad \sqrt[3]{\sqrt{(p/3)^3 + (q/2)^2} + (q/2)} \;-\; \sqrt[3]{\sqrt{(p/3)^3 + (q/2)^2} - (q/2)}$$

and for our example $p = 3/2$ and $q = y/2$. Therefore the formula for the inverse function is

$$x \quad = \quad \sqrt[3]{\sqrt{(1/2)^3 + (y/4)^2} + (y/4)} \;-\; \sqrt[3]{\sqrt{(1/2)^3 + (y/4)^2} - (y/4)} \;. \blacksquare$$

(b) By the Binomial Series Theorem, if a is not a nonnegativee integer and $|y| < 1$, then we have

$$\sum_{k=0}^{\infty} \binom{a}{k} y^k \quad = \quad (1+y)^a$$

so if we set $y = -x^4$ and $a = -\frac{1}{2}$ this becomes

$$\sum_{k=0}^{\infty} \binom{-1/2}{k} (-1)^k x^{4k} \quad = \quad (1-x^4)^{-1/2}$$

provided $|x| < 1$. We may now perform term by term integration to conclude that

$$\int_0^x \frac{dt}{\sqrt{1-t^4}} \quad = \quad \sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{(-1)^k x^{4k+1}}{4k+1} \;.$$

Therefore the series starts off as $Ax + Bx^5 + Cx^9 + Dx^{13}$ where $A = 1$,

$$B \;=\; \frac{1}{10} \qquad C \;=\; \left(\frac{1}{2}\cdot\frac{3}{2}\right)\cdot\frac{1}{9} \;=\; \frac{1}{12} \qquad D \;=\; \frac{15}{8}\cdot\frac{1}{13} \;=\; \frac{15}{104} \;. \blacksquare$$

4. [*25 points*] Let L_0 and L_1 be the vertical lines in the coordinae plane defined by $x = 0$ and $x = 1$ respectively. Find the locus (or set) of all points P in the coordinate plane such that the (shortest) distance from P to L_0 is the **square** of the (shortest) distance from P to L_1, and draw a sketch of the locus.

SOLUTION

If $P = (x, y)$, then the square of the distance from P to L_0 equals x^2 and the distance from P to L_1 equals $|x - 1|$. Therefore the defining equation of the set is $x^2 = |x - 1|$. It will be convenient to split this into a pair of equations:

$$x^2 \;=\; x \;-\; 1 \qquad \text{or} \qquad x^2 \;=\; 1 \;-\; x$$

The first equation can be rewritten in the form

$$\left(x - \frac{1}{2}\right)^2 \;=\; -\frac{3}{4}$$

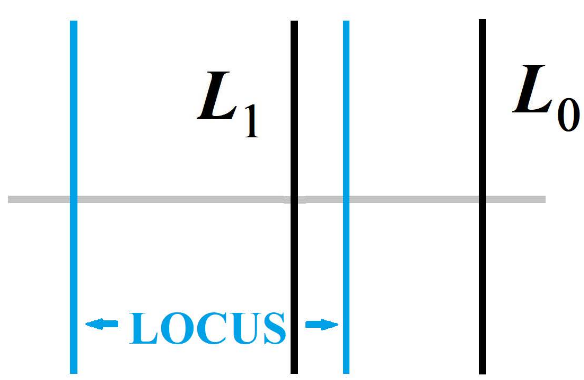
and the second can be rewritten in the form

$$\left(x + \frac{1}{2}\right)^2 \;=\; \frac{5}{4}\ .$$

There are no real number solutions to the first equation because a sum of real squares is never a negative real number, and the solutions to the second are given by

$$x \;=\; \frac{-1 \pm \sqrt{5}}{2}$$

and no conditions whatsover on y. Therefore the locus is equal to a pair of vertical lines. Since one value of x is positive and one is negative, one of the lines lies between L_0 and L_1, while the other lies to the left of L_0.■



5. [*30 points*] For **TEN** of the contributors to the history of mathematics listed below, give the century in which he or she was active. Extra credit may be obtained if more than 10 are answered correctly.

Al-Hazen

Al-Kashi

Bhaskara (a.k.a. Bhaskara II)

Bombelli

Brahmagupta

Fibonacci

Heron (a.k.a. Hero)

Oresme

Pappus

Seki

Thabit ibn Qurra

Viète

SOLUTION

Al-Hazen — 11^{th} century
Al-Kashi — 15^{th} century
Bhaskara (a.k.a. Bhaskara II) — 12^{th} century
Bombelli — 16^{th} century
Brahmagupta — 7^{th} century
Fibonacci — 13^{th} century
Heron (a.k.a. Hero) — 1^{st} century
Oresme — 14^{th} century
Pappus — 4^{th} century
Seki — 17^{th} century
Thabit ibn Qurra — 9^{th} century
Viète — 16^{th} century

6. [*20 points*] (a) State whether the assertion

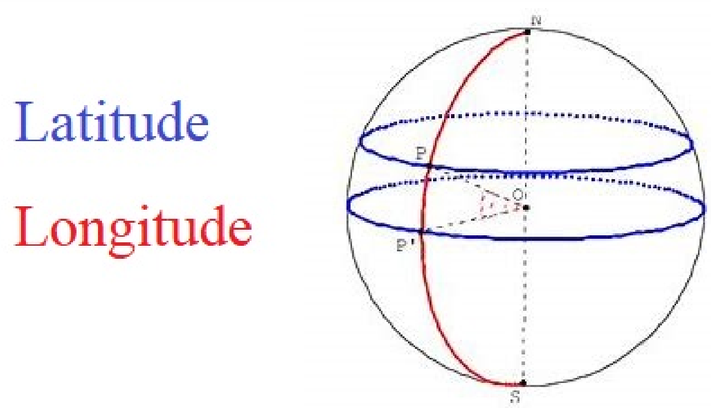
On a sphere, a $45°$ *latitude circle is a great circle*

is true or false and give reasons for your ahswer.

(b) Name two logical issues which were either overlooked in Euclid's *Elements* or were done particularly well, and give reasons for your answer. You may take two issues of one kind or one issue of each kind, and extra credit is possible if three or four issues are properly identified.

SOLUTION

(a) The statement is **FALSE.** By definition a great circle on a sphere is a circle whose center coincides with the center of the sphere. In particular, the length of a great circle is the circumference of the sphere. Neither of these is true for the latitude circle. For a sphere of radius r, the center of this circle is a point which is $r/\sqrt{2}$ from the center of the sphere, and the length of the latitude is $1/\sqrt{2}$ times the circumference of the sphere.



In the drawing above, the radius of the latitude through P is given by $r \cos \angle OPP'$, where $|\angle OPP'|$ is the latitude and r is the radius of the sphere.

(b) Here are some issues which were overlooked in Euclid's *Elements* (this list is not exhaustive):

- In mathematics it is not possible to define everything in terms of other concepts; the chain of definitions must begin somewhere and certain concepts must be taken as primitive or undefined. For example, this applies to the notions of point, lines and planes in classical Euclidean geometry.
- Euclid frequently discusses concepts like two points lying on the same or opposite sides of a line, one point on a line being between two others on the same line, a point lying in the interior (region) of an angle or triangle, and so forth. None of these notions is described formally in the *Elements*, and the corresponding lack of postulates about the properties of these concepts leaves some of the arguments incomplete.

- Another issue in the Elements is its use of the so called principle of superposition which suggests that one can freely move objects without changing their sizes or shapes. Apparently Euclid himself was uncomfortable with the idea of proof by superposition, which was used to prove the Side Angle Side Congruence Theorem for triangles.
- Yet another unstated assumption is that circles and certain other geometrical figures have common points. Surprisingly, the first logical difficulty in the *Elements* is the need for such an assumption, and it appears in the proof of the very first result.
- The lengthy discussion of irrational numbers was later seen to be inadequate (*e.g.*, in the work of Fibonacci on the roots of cubic polynomials).

And here are some issues which were handled particularly well in Euclid's *Elements* (this list is not exhaustive):

- The single most important aspect of the *Elements* is its logical organization, which begins with definitions for important concepts, formulates some basic properties of these concepts that will be assumed, and uses deductive logic to prove new conclusions or theorems. These results were presented in order, with each proof aimed at depending only on the results and assumptions that had been previously established. Euclids logical framework for geometry is very clear, concise and powerful, and it was the default setting for nearly 2000 yeaars.■
- Euclids formulations of various mathematical results quickly superseded some earlier ones and became established as the standard ones in many cases. There are also many instances where his formulations and arguments are still the preferred ones today, not for any sentimental value they may have but simply because they are still the most clear and direct ways to consider certain topics.■
- The treatment of proportionality, essentially due to Eudoxus, was relatively sophisticated for its time and gave a rigorous proof of the basic theorems on this subject when the ratios are irrational numbers.■
- The inclusion of the Fifth Postulate, which is relatively complicated with respect to the other assumptions, turned out to be extremely important and in fact logically independent of the other assumptions which either appeared in the *Elements* or were tacitly assumed. It is ironic that many considered this to be a shortcoming over a period of nearly 2000 years; the vindication for Euclid is that the Fifth Postulate is logically indispensable rather than logically redundant.■