Solution to Additional Exercise

02.4

Follow the hint, and take $p=(0,0)$, $q=(a,0)$ where $a>0$. Then the (square of the) distance equation is

$$x^2+y^2=r^2\left((x-a)^2+y^2\right)$$

which can be rewritten as

$$(1-r^2)x^2+2r^2ax-r^2a^2+(1-r^2)y^2=0.$$

Divide this by $(1-r^2)$:

$$\left(x^2+\frac{2r^2a}{1-r^2}x\right)+y^2=\frac{r^2a^2}{1-r^2}$$

Complete the square of the x-term:

$$\left(x^2+2\frac{r^2a}{1-r^2}x+\frac{r^4a^2}{(1-r^2)^2}\right)+y^2=\frac{r^4a^2}{(1-r^2)^2}+\frac{r^2a^2}{1-r^2}$$

$$=\frac{r^4a^2+r^2a^2(1-r^2)}{(1-r^2)^2}=\frac{r^2a^2}{(1-r^2)^2}$$

Note that the right side is positive since $1-r^2>0$.

The equation above defines a circle with center $\left(\frac{-r^2a}{1-r^2},0\right)$ and radius $\sqrt{\frac{r^4a^2+r^2a^2(1-r^2)}{(1-r^2)^2}}$

$$=\frac{ra}{(1-r^2)}$$

<u>Note</u> If we let $r \to 1$ the radius goes to $+\infty$, and the circle "flattens out" into the perpendicular bicector of the closed line segment joining $(0,0)$ to $(a,0)$.