Math 153
Spring 2020
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SOLUTIONS TO EXERCISES FROM `math153exercises07a.pdf`

21. (a)We shall use the identity

$$d^2 \;+\; \left(\frac{d^2-1}{2}\right)^2 \;=\; \left(\frac{d^2+1}{2}\right)^2 .$$

If d is odd, then the second quantity in the expression is even (note that d^2-1 is divisible by 4) and the second quantity is odd (in this case $d^2+1 \equiv 2 \bmod 4$). Define a sequences of numbers d_k as follows, starting with $d_1 = 3$, $c_2 = 4$, and

$$d_{k+1} \;=\; \frac{d_k^2+1}{2}\,, \qquad c_{k+1} \;=\; \frac{d_{k+1}^2-1}{2} \;=\; d_{k+1}-1$$

for $k \geq 2$. Then by induction and the previous identity we have

$$d_k^2 \;+\; c_{k+1}^2 \;=\; d_{k+1}^2$$

so that by induction we also have

$$d_1^2 \;+\; \sum_{n=1}^{k} c_{n+1}^2 \;=\; d_{k+1}^2$$

showing that the sequence $\{d_1, c_2, \cdots, c_{k+1}\}$ is a Pythagorean $(k+1)$-tuple for each k.■

(b) Probably the easiest way to construct an infinite family of examples is to take each sequence $\{d_k\}$ and multiply everything by a fixed integer $m \geq 2$. However, there are also methods for constructing many other families of examples. In particular, if we start with $1^2 + 2^2 + 2^2 = 3^2$ we can start with that sequence and use $3^2 + 4^4 = 5^2$ to obtain $1^2 + 2^2 + 2^2 + 4^4 = 5^2$, then use $5^2 + 12^2 = 13^2$ to obtain $1^2 + 2^2 + 2^2 + 12^2 = 13^2$, and so on. Also, in another direction we can start with $d_1 \geq 7$ as an arbitrary odd integer and proceed as in (a).■

22. By the main result in `history07a.pdf` we know that

$$F_n \;=\; \frac{\phi^n - \psi^n}{\sqrt{5}}$$

where $\phi = \frac{1}{2}(\sqrt{5}+1)$ and $\psi = \frac{1}{2}(\sqrt{5}-1)$. It follows that $\phi > 1 > \psi > 0$. We shall prove the following more general result:

FORMULA. *If* $\alpha > 1 > \beta > 0$, *then*

$$\lim_{n\to\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} \;=\; \alpha .$$

Derivation. Rewrite the quotient on the left hand side as

$$\frac{\alpha^{n+1}(1-\beta^{n+1}\alpha^{-n-1})}{\alpha^n(1-\beta^n\alpha^{-n})} \quad = \quad \alpha \cdot \frac{1-\beta^{n+1}\alpha^{-n-1}}{1-\beta^n\alpha^{-n}} \ .$$

Our assumptions imply that $1 > \beta\alpha^{-1} > 0$ and therefore both $1-\beta^n\alpha^{-n}$ and $1-\beta^{n+1}\alpha^{-n-1}$ go to 1 as $n \to \infty$. Therefore the limit of the original fraction is α. If we specialize to the ratio F_{n+1}/F_n, this formula shows that the limit of the latter equals ϕ.∎

23. (a) If $a > 1$ is abundant, then $a < \sum_{d<a, d|a} d$, so that

$$ma \quad < \quad m \cdot \sum_{d<a,d|a} d \quad = \quad \sum_{d<a,d|a} md \quad \leq \quad \sum_{c|c<ma,c|ma} c$$

where the final inequality holds because the numbers md are some of the proper divisors of ma. In fact there is always at least one more given by $c = 1$. These inequalities show that ma is abundant. ∎

(b) If $a > 1$ is perfect, then $a < \sum_{d<a,d|a} = d$, so that

$$ma \quad = \quad m \cdot \sum_{d<a,d|a} d \quad = \quad \sum_{d<a,d|a} md$$

and as in the first part, we know that 1 is yet another proper divisor of md. Therefore the right hand side is strictly less than

$$1 \ + \ \sum_{d<a,d|a} md \quad \leq \quad \sum_{c|c<ma,c|ma} c$$

and thus we once again conclude that ma is abundant.∎

(b) Suppose that $a > 1$ is perfect and $d \neq 1$ properly divides a, so that $a = md$ for some $m > 1$. By definition we know that d is perfect, abundant or deficient. If either of the first two possibilities holds then by the first two parts of the exercise we know that $a = md$ is abundant. Since this is not the case it follows that d must be deficient.∎

24.

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25.

∎ See math153solutions07aa.pdf for these problems.

26.

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27.