***Source:* Posamentier and Lehmann, *Those (Fabulous) Fibonacci Numbers*, Appendix B.**

8. (p. 42) We have, by factoring (difference of squares),

$$
\begin{aligned}
&F_k^2 - F_{k-2}^2 \\
&= (F_k - F_{k-2})(F_k + F_{k-2}) \\
&= F_{k-1}(F_k + F_{k-2}) \\
&= F_{k-1}F_k + F_{k-1}F_{k-2} \\
&= F_{k-1}F_{k-2} + F_kF_{k-1}
\end{aligned}
$$

First we'll prove the following lemma, which will come in useful and later on.

LEMMA. $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$

Proof of Lemma

We proceed by induction on n. (Actually, this will be a form of induction that is called "strong induction," where we will be using the statements for $n = k - 1$ and for $n = k$ to prove the statement for $n = k + 1$. This will also mean that we will need two base cases rather than just one; so we will have to check the statement for $n = 1$ and for $n = 2$.) For $n = 1$, we have to check that $F_{m+1} = F_{m-1}F_1 + F_mF_2$, or, since $F_2 = 1$ and $F_2 = 1$, we have to check that $F_{m+1} = F_{m-1} + F_m$; this is, of course, true because it is the very relation by which we define the Fibonacci numbers. For $n = 2$, we have to check that $F_{m+2} = F_{m-1}F_2 + F_mF_3$, or, since $F_2 = 1$ and $F_3 = 2$, we have to check that $F_{m+2} = F_{m-1} + 2F_m$; and this is true by the following sequence of equalities:

$$F_{m-1} + 2F_m = (F_{m-1} + F_m) + F_m = F_{m+1} + F_m = F_{m+2}$$

Now assume the statement is true for $n = k - 1$ and $n = k$, that is, assume

$$F_{m+k-1} = F_{m-1}F_{k-1} + F_mF_k$$

and

$$F_{m+k} = F_{m-1}F_k + F_mF_{k+1}$$

(This is our induction hypothesis.) Then

$$\begin{aligned}
&F_{m-1}F_{k+1} + F_mF_{k+2} \\
&= F_{m-1}(F_{k-1} + F_k) + F_m(F_k + F_{k+1}) \\
&= F_{m-1}F_{k-1} + F_{m-1}F_k + F_mF_k + F_mF_{k+1} \\
&= F_{m-1}F_k + F_mF_{k+1} + F_{m-1}F_{k-1} + F_mF_k \\
&= F_{m+k} + F_{m+k-1} \\
&= F_{m+k+1}
\end{aligned}$$

That is, $F_{m-1}F_{k+1} + F_mF_{k+2} = F_{m+k+1}$, which is exactly the statement for $n = k + 1$. So our induction is complete.

Now using this lemma, we see (by substituting $m = k$ and $n = k - 2$) that

$$F_{k-1}F_{k-2} + F_kF_{k-1} = F_{k+k-2} = F_{2k-2}$$

Hence $F_k^2 - F_{k-2}^2 = F_{2k-2}$, and the proof is complete.

9. (p. 43) $F_n^2 + F_{n+1}^2 = F_{2n+1}$; that is, the sum of the squares of the Fibonacci numbers in positions n and $n + 1$ (consecutive positions) is the Fibonacci number in place $2n + 1$.

Proof

Using the lemma from item 8 (above), we prove our original statement. In the lemma, let $m = n + 1$ and $n = n$. Then we get $F_{2n+1} = F_nF_n + F_{n+1}F_{n+1}$, or, in other words, $F_{2n+1} = F_n^2 + F_{n+1}^2$, which is the equation we wanted.

10. (p. 43) A proof that $F_{n+1}^2 - F_n^2 = F_{n-1} \cdot F_{n+2}$

It follows immediately, because of the binomial theorem $\left[a^2 - b^2 = (a+b)(a-b)\right]$ and the definition of the Fibonacci numbers, that

$$F_{n+1}^2 - F_n^2 = (F_{n+1} + F_n)(F_{n+1} - F_n) = F_{n+2} \cdot F_{n-1}$$

11. (p. 44) A proof of $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$, where n ≥ 1.
By mathematical induction:

For $n = 1$:

$$P(1): \quad F_0 F_2 = F_1^2 + (-1)^1;$$
$$0 \cdot 1 = 1^2 - 1 = 0$$

For $n = k$:

$$P(k): \quad F_{k-1}F_{k+1} = F_k^{\,2} + (-1)^k, \text{ where } k \geq 1.$$

For $n = k + 1$:

$$P(k+1): \text{ to check if it is true for } k+1:$$

$$\begin{aligned} F_{k+2}^2 - F_{k+1}^2 &= (F_k + F_{k+1})^2 - F_{k+1}^2 \\ &= F_k^2 + 2F_{k+1}F_k + F_{k+1}^2 - F_{k+1}^2 \\ &= F_k(F_k + 2F_{k+1}) \\ &= F_k\left[(F_k + F_{k+1}) + F_{k+1}\right] \\ &= F_k(F_{k+2} + F_{k+1}) \\ &= F_k \cdot F_{k+3} \end{aligned}$$

And we will complete the proof by mathematical induction if we in the following use the assumption for $n = k$ in the form $F_{k-1} \cdot F_{k+1} - F_k^2 = (-1)^{k+1.}$

$P(k+1)$: To check if $P(k+1)$ is true: $F_k F_{k+2} = F_{k+1}^2 + (-1)^{k+1}$

$$\begin{aligned}F_k F_{k+2} - F_{k+1}^2 &= F_k(F_{k+1} + F_k) - F_{k+1}^2 \\ &= F_k F_{k+1} + F_k^2 - F_{k+1}^2 \\ &= F_k^2 + F_{k+1}(F_k - F_{k+1}) \\ &= F_k^2 + F_{k+1}(-F_{k-1}) \\ &= F_k^2 - F_{k+1}F_{k-1} \\ &= (-1)(F_{k+1}F_{k-1} - F_k^2) \\ &= (-1)(-1)^k \\ &= (-1)^{k+1}\end{aligned}$$

Therefore, $P(k+1)$: $F_k F_{k+2} - F_{k+1}^2 = (-1)^{k+1}$

or $F_k F_{k+2} = F_{k+1}^2 + (-1)^{k+1}$

$= (-1)^k$

12. (p. 47) We have to show that F_{mn} is divisible by F_m for any n and m. We proceed by induction on n.

For $n = 1$, we need to check that $F_{1\cdot m}$ is divisible by F_m; and this is, of course, true. Now assume F_{mp} is divisible by F_m (this is our induction hypothesis—the statement for $m = p$). We want to prove the statement for $m = p + 1$, that is, that $F_{m(p+1)}$ is divisible by F_m. Using the Lemma in number 8, we get:

$$F_{m(p+1)} = F_{mp+m} = F_{mp-1}F_m + F_{mp}F_{m+1}$$

which is divisible by F_m because both summands (addends) $F_{mp-1}F_m$ and $F_{mp}F_{m+1}$ are divisible by F_m (by the induction hypothesis, F_{mp} is divisible by F_m), so it is also $F_{m(p+1)}$. This completes the induction.

14. (p. 49) We again proceed by induction. For $n = 1$, we need to check that $L_1 = L_3 - 3$; and this is, of course, true