SOLUTIONS

1. $$\frac{5}{13} = \frac{1}{\frac{13}{5}} = \frac{1}{2+\frac{3}{5}} = \frac{1}{2+\frac{1}{5/3}} =$$

$$\frac{1}{2+\frac{1}{1+\frac{2}{3}}} = \frac{1}{2+\frac{1}{1+\frac{1}{3/2}}} =$$

$$\frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}}. \blacksquare$$

2. $$\frac{1}{xy} = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy}, \text{ so } x+y=1$$

Solutions $x = \frac{p}{q}$ $(p<q)$, $y = \frac{q-p}{q}$. $\blacksquare$

3. $x + y = \frac{x}{y}$. Clear fractions

$$xy + y^2 = x, \text{ or } y^2 + xy - x = 0$$

So $$y = \frac{-x \pm \sqrt{x^2+4x}}{2}.$$

This will give a solution with $x, y > 0$ rational if we take the + option and we have that

$$x^2 + 4x \quad \text{is a perfect square.}$$

So we need to solve $x^2 + 4x = z^2$ with $x > 0$ rational.

Let $z = x + d$ $(d > 0)$. Then

$$x^2 + 4x = x^2 + 2dx + d^2, \text{ so}$$
$$4x - 2dx = d^2, \text{ or } x = \frac{d^2}{4 - 2d}.$$

This is positive if $d < 2$, so there are infinitely many valid choices of solutions x, y. ∎

4. $x + y = A \Rightarrow A^2 = (x+y)^2 = x^2 + 2xy + y^2 = B + 2xy$

Hence $xy = \frac{A^2 - B}{2}$. This has a solution with $x, y \geq 0$ if $A^2 > B$. [redacted]

Now x, y must also satisfy $x+y=A$, which yields the equation

$$x(A-x) = \frac{A^2-B}{2} \quad \text{or}$$

$$(A^2-B) - 2xA + 2x^2 = 0.$$

So we need to understand when this equation has a positive rational root. Use the Quadratic Formula

$$x = \frac{2A \pm \sqrt{4A^2 - 8(A^2-B)}}{4} =$$

$$\frac{2A \pm \sqrt{8B - 4A^2}}{4}$$

So positive rational solutions exist $\Leftrightarrow$

$2B - A^2$ is a perfect square in the rationals. Summarizing, the two conditions are

$$2B > A^2 > B \quad \text{and } 2B - A^2 \text{ is a square.}$$

<u>Example</u> $A=12$ $B=80$

$$160 > 144 > 80 \quad \text{and}$$

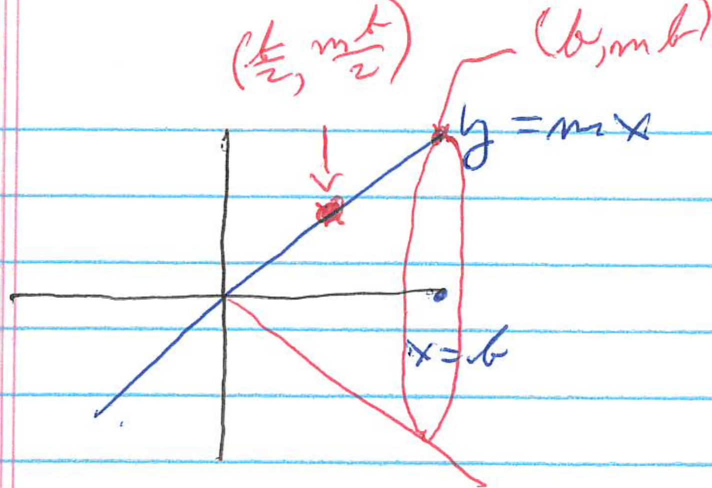
$$160 - 144 = 16 = 4^2$$

There are other possible choices for A and B. Try to find ~~the~~ a few.

Hint: If we let $2B - A = C^2$, this amounts to finding A and C so that $A^2 + C^2$ is even and $A^2 > \frac{1}{2}(A^2 + C^2)$.

Question: For which A, B do the equations have only one solution, and what does this mean geometrically?

5.



By Pappus, the surface area is

(length of segment) $\cdot\, 2\pi \cdot$ (y-coord of its centroid)

The length is $b\sqrt{1+m^2}$, and the centroid is $\frac{1}{2}(b, mb)$. Substituting, we get

$$S = b\sqrt{1+m^2}\cdot 2\pi \cdot \frac{mb}{2} =$$

$$\pi b^2 m\sqrt{1+m^2}. \blacksquare$$

6. $x = 2r+1$ & $x \equiv 2\,(3) \Rightarrow$ ~~2r~~

$2r+1 \equiv 2\,(3) \Rightarrow 2r \equiv 1\,(3) \Rightarrow$

$r \equiv 2\,(3) \Rightarrow r = 3s+2$, so

$x = 6s+5$ ~~[illegible]~~ Now $x \equiv 3\,(5)$

$\Rightarrow 6s+5 \equiv 3\,(5) \Rightarrow s \equiv 3\,(5) \Rightarrow$

$s = 5t+3 \Rightarrow x = 6s+5 = 30t+23.$

Finally, $x \equiv 5(7) \Rightarrow 30t + 23 \equiv 5(7)$

$\Rightarrow 2t + 2 \equiv 5(7) \Rightarrow 2t \equiv 3(7)$.

Now $2 \times 4 \equiv 1(7)$, so the last congruence becomes $t = 4 \cdot 2t = 4 \cdot 3 \equiv 5(7)$, so that $t = 7u + 5$ and hence

$x = 30t + 23 = 30(7u+5) + 23 =$

$210u + 150 + 23 = 210u + 173$. $\blacksquare$