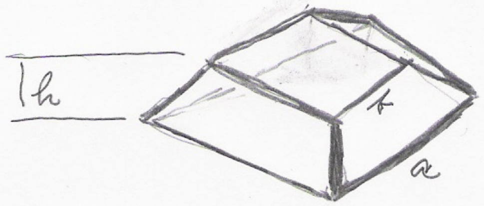
Frustrum Volume Formula

(Burton, p. 56)

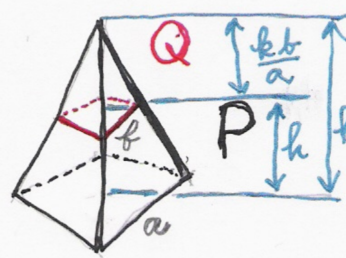


$$V = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

Derivation from the volume formula for a pyramid:

Let P be the pyramid whose base is a square of side a, and let Q be the smaller pyramid whose base is a square of side b. Then the volume of the frustrum is $\mathrm{vol}(P) - \mathrm{vol}(Q)$.

Let k be the altitude of P. Then $\frac{kb}{a}$ is the altitude of Q, so that



$$\mathrm{vol}(P) = \frac{1}{3} a^2 k$$

$$\mathrm{vol}(Q) = \frac{1}{3} b^2 k \frac{b}{a}$$

We need to express k in terms of h. Now

$$k = \frac{kb}{a} + h, \text{ so that}$$

$$h = k\left(1 - \frac{b}{a}\right) = k\left(\frac{a-b}{a}\right) \text{ and } k = \frac{ha}{a-b}.$$

Using these, we may rewrite the volumes as

$$\mathrm{vol}(P) = \frac{1}{3} a^2 \frac{ha}{a-b} = \frac{h}{3} \frac{a^3}{a-b}$$

$$\mathrm{vol}(Q) = \frac{1}{3} b^2 \frac{ha}{a-b} \cdot \frac{b}{a} = \frac{h}{3} \frac{b^3}{a-b}.$$

Hence the frustrum volume is given by the difference

$$\frac{h}{3}\left[\frac{a^3}{a-b} - \frac{b^3}{a-b}\right] = \frac{h}{3} \frac{a^3-b^3}{a-b} =$$

$$\frac{h}{3}(a^2 + ab + b^2)$$

which is the formula we want. ∎