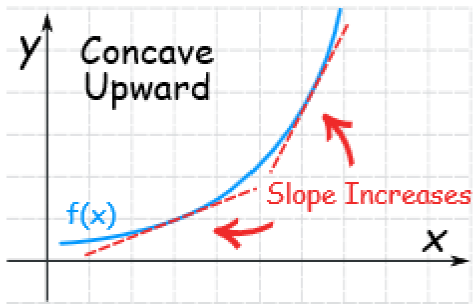
1.C. The False Position Method

Versions of the Method of False Position, which gives successive approximations converging to a solution for an equation of the form **f(x) = 0,** were known and used in ancient Egyptian and Babylonian mathematics. Later and more sophisticated versions provide a means for studying more complicated equations, and the objective here is to describe the enhanced versions needed to develop such procedures by using the method to find roots quadratic and cubic equations. For the sake of simplicity we shall first restrict attention to functions which are strictly increasing and convex (the first and second derivatives are positive; this is the first year calculus concept of ***concave upward***).



Source: **https://www.mathsisfun.com/calculus/concave-up-down-convex.html**

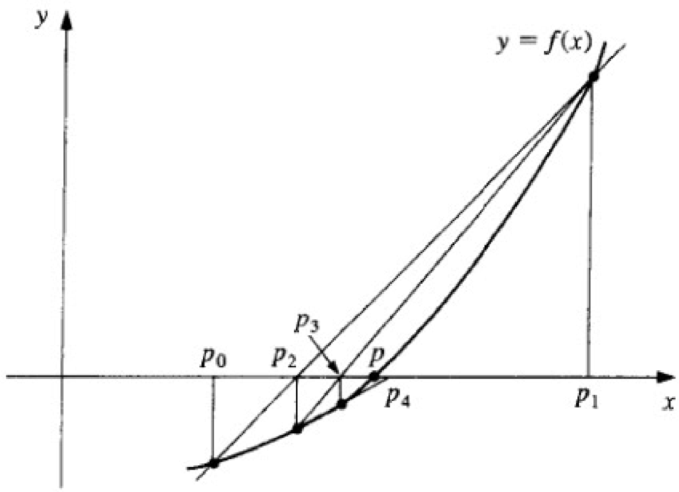
Suppose now that we have a reasonable function **f(x)** and there are two points **a** and **b** such that **f(a)** and **f(b)** are nonzero with different signs, with exactly one value of **r** between **a** and **b** such that **f(r) = 0.** One naïve way to find an approximation for a root (a **trial root**) is to ***pretend that the function is linear between* a *and* b *and to find the value* c *for which this linear function's value is zero.*** In graphical terms, **c** is the *x* – intercept of the line between **(a, f(a))** and **(b, f(b)).** To find **c,** we first need to write out the equation for the line joining **(a, f(a))** and **(b, f(b)):**

$$y = f(b) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-b)$$

If we set **y = 0** and solve for **x,** we obtain the *x* – intercept, which is **c:**

$$c = b - \left(\frac{b-a}{f(b)-f(a)}\right)f(b)$$

The next step is to evaluate **f(c)** and determine whether it is positive or negative or zero. If it is zero, then obviously we are done; otherwise, we let **c** replace either **a** or **b,** taking the one for which the sign of **f(c)** is equal to the sign of **f(a)** or **f(b).** The effect of this replacement is to narrow the search for the root to a slightly smaller interval. We can now repeat this procedure on the smaller interval, obtaining another, even smaller interval, and so forth. For convex functions it turns out that the sign of **f(c)** is equal to the sign of **f(a).** It is not all that difficult to prove this assertion (see **http://math.ucr.edu/~res/math153-2019/history01d.pdf** for details), but here we shall be content to give a picture illustrating the situation. In this drawing **a, b** and **c** correspond to $\mathbf{p_0}$, $\mathbf{p_1}$ and $\mathbf{p_2}$ respectively.



Source: **http://jonathanmetodos.blogspot.com/2010/07/method-of-false-position.html**

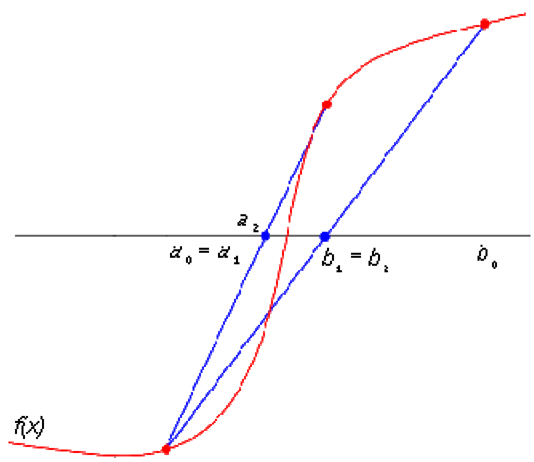
The picture now suggests the next approximation. The function is negative at $\mathbf{p_2}$ and positive at $\mathbf{p_1}$, so find another trial root $\mathbf{p_3}$ by assuming the function is linear between $\mathbf{p_2}$ and $\mathbf{p_1}$. Once again the value of **f** at $\mathbf{p_2}$ is negative by the convexity of **f,** so repeat this procedure for the interval from $\mathbf{p_3}$ to $\mathbf{p_1}$. Eventually the $\mathbf{p_k}$ values accumulate to some limit **L,** and for this limiting value we have **f(L) = 0.** Here are two examples, the first of which is for the square root of **2** and the second of which is for the root of a less tractable cubic equation; in fact, we shall see that the question of finding and estimating roots for cubic equations has played a major role in the development of mathematics ever since ancient times.

LX	LY	RX	RY	TRIAL	f(TRIAL)	
1	-1	2	2	1.33333333	-0.222222222	**f(x) = x^2-2**
1.33333333	-0.222222222	2	2	1.4	-0.04	
1.4	-0.04	2	2	1.41176471	-0.006920415	
1.41176471	-0.006920415	2	2	1.4137931	-0.001189061	
1.4137931	-0.001189061	2	2	1.41414141	-0.000204061	
1.41414141	-0.000204061	2	2	1.41420118	-3.50128E-05	
1.41420118	-3.50128E-05	2	2	1.41421144	-6.00729E-06	
1.41421144	-6.00729E-06	2	2	1.4142132	-1.03069E-06	
1.4142132	-1.03069E-06	2	2	1.4142135	-1.76838E-07	
1.4142135	-1.76838E-07	2	2	1.41421355	-3.03407E-08	
1.41421355	-3.03407E-08	2	2	1.41421356	-5.20563E-09	
1.41421356	-5.20563E-09	2	2	1.41421356	-8.93146E-10	

Explanation of columns: At each step the left and right hand points of the interval are **LX** and **RX,** and the ordered pairs **(LX,LY)** and **(RX,RY)** denote the corresponding points on the graph of the function f. The column **TRIAL** gives the approximation of the root corresponding to linear approximation over the interval, and **f(TRIAL)** is the value of **f** at **TRIAL.** Note that the numbers in the first column appear to converge to a limit, and this limit is the desired root of **f.** This is illustrated by the fact that the corresponding numbers in the last columns go to zero. The procedure for computing the successive approximations is described in the file **http://math.ucr.edu/~res/math153-2019/history01c.program.pdf**.

LX	LY	RX	RY	TRIAL	f(TRIAL)	
0	-1	1	1	0.5	-0.375	**f(x) = x^3+x-1**
0.5	-0.375	1	1	0.63636364	-0.105935387	
0.63636364	-0.105935387	1	1	0.67119565	-0.026428288	
0.67119565	-0.026428288	1	1	0.67966165	-0.006375484	
0.67966165	-0.006375484	1	1	0.68169102	-0.001525358	
0.68169102	-0.001525358	1	1	0.68217582	-0.000364224	
0.68217582	-0.000364224	1	1	0.68229153	-8.6928E-05	
0.68229153	-8.6928E-05	1	1	0.68231915	-2.07444E-05	
0.68231915	-2.07444E-05	1	1	0.68232574	-4.9503E-06	
0.68232574	-4.9503E-06	1	1	0.68232731	-1.1813E-06	
0.68232731	-1.1813E-06	1	1	0.68232769	-2.81893E-07	

If the function is increasing but not convex, then one must modify this procedure slightly. It will be enough to describe what happens at the first step. In general we need to check first whether the sign of f(c) is positive or negative (if the value is zero we have found the root!). In this case we let **c** replace one of the endpoints **a** and **b,** taking the one for which the sign of **f(c)** is equal to the sign of **f(a)** or **f(b)** respectively**.** Once again the effect of this replacement is to narrow the search for the root to a slightly smaller interval**,** and under favorable conditions we get a sequence whose limit is the desired root of **f.** Here is an illustration of a case where **f(c)** is positive and in fact one eventually shrinks the interval at both ends**:**



(*Source:* http://en.wikipedia.org/wiki/False_position_method)

In addition to the references cited in Unit 1 of the notes, some additional information and comments about this method appear on pages 73 – 75 of the following standard textbook on numerical methods**:**

> **R. L. Burden and J. D. Faires,** *Numerical Analysis* (9th Ed.)**.** Brooks – Cole**,** Boston**,** 2011**. ISBN – 10:** 0 – 538 – 73351 – 9**.**

Also**,** here is an online reference which includes a proof that the method's iterated approximations actually converges to the desired solution**:**

http://astro.temple.edu/~dhill001/course/numanalfall2009/False%20Position%20Section%202_2.pdf