1.E. Egyptian vs. modern multiplication algorithms

As noted in the file `egyptian-addition.pdf`, the Egyptian and modern algorithms for addition are basically equivalent. Since the same is not true for multiplication, it is illuminating to compare the algorithmic processes for positive integer multiplication in Egyptian and modern mathemtics. Let P and Q denote the two numbers, with $0 < P \leq Q$.

The Egyptian method. The basic idea is to use repeated doubling of Q and the base 2 expansion of P to find the product. Hence the first step is to express P as a sum $p_0 + 2p_1 + ... + 2^m p_m$ where each $p_i = 0$ or 1. One then computes $2^k Q$ recursively by $2^k Q = 2^{k-1}Q + 2^{k-1}Q$. Finally, one finds the product by the identity

$$P \cdot Q \quad = \sum_{k \text{ s.t. } p_k = 1} 2^k Q \ .$$

One can view multiplication of integers as repeated addition, and the latter is done in two stages, one in finding the powers of two times Q and the other in finding the sum of the appropriate numbers. Some work is also needed to find the base 2 expression of P since Egyptian numbers were an early (incomplete) form of our current numeration system (no zero, an upper limit on what could be expressed in their notation). The upper limit of ten million meant that one always had $P < 2^{12} = 4096$, so the amount needed to find the base 2 expression can be done in a bounded number of steps.

The modern method. Here we use the base 10 expressions for both factors:

$$P \quad = \quad \sum_{i=0}^{L} a_i 10^i \qquad Q \quad = \quad \sum_j b_j 10^j \qquad \text{where} \quad 0 \leq i, j \leq 9$$

The algorithm then breaks into two pieces, the first for computing products such that $0 \leq P \leq 9$ and the second to take the results for each $P \cdot b_j$ and form the sum

$$P \cdot Q \quad = \quad \sum_j P \cdot b_j \cdot 10^j \ .$$

For each $P \cdot b$ as above, we proceed as follows:

INITIALLY SET $m_i = a_i$
FOR i STARTING AT 0 : DO THE FOLLOWING
WRITE THE PRODUCT $m_i \cdot b$ AS $10 \cdot d + c_i$ $(\mathbb{LOOP})$
DEFINE NEW m_j FOR $j > i$ VIA NEW $\sum m_j 10^{j-i}$:= OLD $10 \cdot d + \sum m_j 10^{j-i}$
REPLACE i BY $i + 1$, REPEAT $\mathbb{LOOP}$ UNTIL $i = L$
DISPLAY THE PRODUCT AS $\sum_i c_i \cdot 10^i$ AND STOP

Both methods have hidden features. In the Egyptian method, clearly one needs a good means for finding the base 2 expansion of the smaller factor, while in the modern method one tacitly assumes that multiplication of single digit numbers is known and easy to access from memory. The modern algorithm is more complicated in several respects, but it can also be used more generally and does not require the use of two different number system bases.