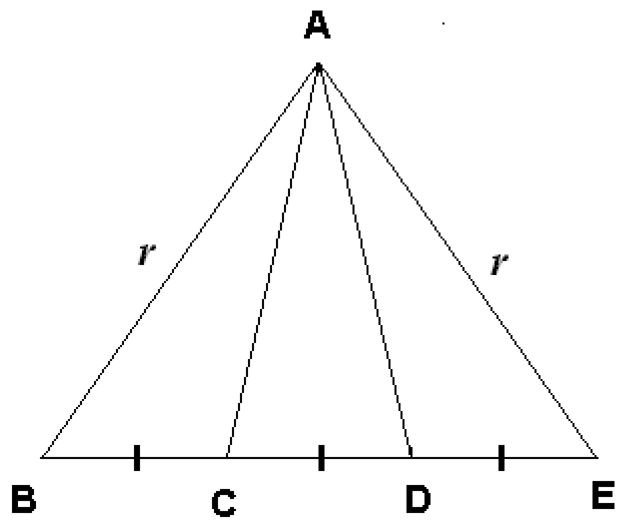
2.C. An Easy Trisection Fallacy

We have already noted that any purported straightedge and compass construction for trisecting an angle will be incorrect**.** The following simple example illustrates how appealing such a construction might appear at first and how one can look more closely to find a mistake**.**

Suppose we are given an angle $\angle$ **BAE** as in the diagram below and we wish to trisect it**.** Let's assume that the lengths of the segments **[BA]** and **[AE]** are equal**,** say to r**.** It is known that segments can be divided into any number of pieces of equal length by straightedge and compass**,** so apply this to segment **[BE]** and divide it into three equal segments that we shall call **[BC], [CD]** and **[DE].** If we look at the picture it might seem that the rays **[AC** and **[AD** trisect $\angle$**BAE,** but is this really true**?**



One can use the classical methods of Euclidean geometry to conclude that the segments **[AC]** and **[AD]** have equal length**,** and it is possible to analyze this figure even further using classical methods**,** but we shall take a shortcut using trigonometry**.**

Let h denote the common altitude of the isosceles triangles $\triangle$**BAE** and $\triangle$**CAD,** and let **|XY|** denote the length of the segment joining **X** and **Y.** Then standard results in trigonometry imply the following relationships**:**

$$\tan \tfrac{1}{2}\angle \mathrm{BAE} \;=\; |\mathrm{BE}|/2h \qquad\qquad \tan \tfrac{1}{2}\angle \mathrm{CAD} \;=\; |\mathrm{CD}|/2h \;=\; |\mathrm{BE}|/6h$$

From these formulas we conclude that $\tan \tfrac{1}{2}\angle \mathrm{CAD}$ is one third of $\tan \tfrac{1}{2}\angle \mathrm{BAE}$**.** If this construction yielded a trisection then we would have a trigonometric identity of the form

$$(\tan x)/3 \;=\; \tan(x/3)$$

and one can check directly from tables (or a scientific calculator) that the first expression is always greater than the second**.** Since the tangent function is strictly increasing**,** it follows that ***the measure of the middle angle is always larger than the measures of the angles on both sides*****.**

It is also possible to disprove this trisection fallacy using classical methods from Euclidean geometry**,** but the argument is somewhat longer**.**