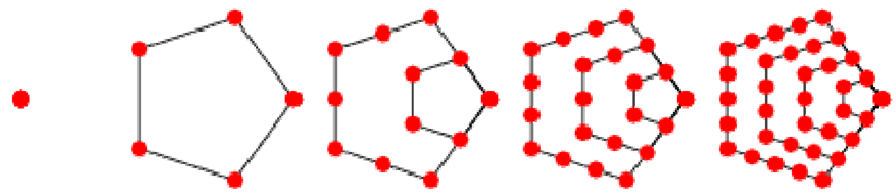
2.E. Polygonal numbers

The Pythagoreans were interested in certain geometrically determined sequences of numbers called polygonal numbers, and the first few cases (***triangular*** and ***square*** numbers) are mentioned on page 95 of Burton. Pentagonal numbers are also mentioned; their definition is suggested by the following picture.



(Source: http://mathworld.wolfram.com/PentagonalNumber.html)

As in the case of triangular and square numbers, if one knows the n^{th} pentagonal number p_n then the next one is given recursively in terms of p_n. One of the exercises for this unit is to find the recursive formula and to derive a closed formula for p_n as an explicit function of n.

Clearly one can proceed indefinitely, starting with hexagonal numbers. The following online references contain further information on this topic:

http://en.wikipedia.org/wiki/Polygonal_number

http://mathworld.wolfram.com/PolygonalNumber.html

One particularly noteworthy result about polygonal numbers is the ***Polygonal Number Theorem***, which was conjectured by P. de Fermat in the 17th century. This result states that, for all $m \geq 3$, every positive integer is a sum of m numbers that are $m-$gonal numbers (a sum of three triangular numbers, four perfect squares, five pentagonal numbers, and so on). Partial results were obtained by several leading mathematicians during the 18th and early 19th centuries, and ultimately the full result was proved by A. L. Cauchy (1789 – 1857). There is a short proof of this result (with a few background references) in the following paper:

Nathanson, Melvyn B. A short proof of Cauchy's polygonal number theorem. *Proc. Amer. Math. Soc.* **99** (1987), 22 – 24.